Relationship between Hamiltonian and Lagrangian perturbation theories

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Motivation

Free theory: the harmonic oscillator

Perturbation theories (no time derivatives)

Perturbation theories (with time derivatives)

Perturbation theories (with two time derivatives)

Motivation

- Perturbative quantum gravity is used, e.g., for studying inflationary universe.
- Lagrangian perturbation theory is used in most calculations.
- In some context one needs to use Hamiltonian perturbation theory.
  - The Hartle-Hawking state for quantum field theory in Schwarzschild spacetime is a thermal state; A thermal state is a statistic ensemble with weight $e^{-H/k_B T}$, where H is the Hamiltonian.
  - To resolve some conceptual issues in perturbative quantum gravity, e.g., infrared divergences in the Faddeev-Popov-ghost propagator, Hamiltonian perturbation theory is useful. (e.g., J. Gibbons, AH, W.C.C. Lima, PRD 103 (2021) 065016).

Motivation

In most textbooks

- The standard formula for Hamiltonian perturbation theory is derived in theories with interaction terms containing no time derivatives.
- Interaction terms in Hamiltonian and Lagrangian perturbation theories are identical if there is no time derivative.
- One keeps using the formula derived in Hamiltonian perturbation theory using the interaction terms in Lagrangian perturbation theory even when there are interaction terms with time derivatives.

BUT...

Motivation

(The following statements will be explained shortly in a simple model.)

(A) The interaction terms are different in Hamiltonian and Lagrangian perturbation theories if they contain time derivatives.

(B) There is a subtle difference in the propagators (the time-ordered products of the quantum variables) between the two perturbation theories.

The logical conclusion is that the two perturbation theories are equivalent despite (A) because of (B).

In this talk I'll explain how this equivalence comes about in Quantum Mechanics of one variable.

Theories with time-derivative interactions

- perturbative quantum gravity:

$$\mathcal{L} \propto \sqrt{-g} R \, .$$

If we let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($\eta_{\mu\nu}$: background metric), there are terms (after integration by parts), e.g., of the form $h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} = h^{00} \partial_t h_{\mu\nu} \partial_t h^{\mu\nu} + \cdots$.

- Scalar QED:

$$\begin{aligned} \mathcal{L} &= (\partial^\mu \phi^* - ieA^\mu \phi^*)(\partial_\mu \phi + ieA_\mu \phi) + \cdots \\ &= ieA^0[(\partial_t \phi^*)\phi - \phi^* \partial_t \phi) + \cdots \, . \end{aligned}$$

Free theory: Harmonic oscillator

The Lagrangian

$$L_{\mathrm{free}} = \frac{m}{2}(\dot{q}^2 - \omega^2 q^2) = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)\,, \quad m = 1.$$

where $\dot{q} = dq/dt$. The canonical conjugate momentum:

$$p = \frac{\partial L_{\mathrm{free}}}{\partial \dot{q}} = \dot{q}\,.$$

The quantization condition (in the Heisenberg picture):
$[q(t), \dot{q}(t)] = i$.

$$q(t) = \frac{1}{\sqrt{2\omega}}(a e^{-i\omega t} + a^\dagger e^{i\omega t})\,, \quad [a, a^\dagger] = 1\,.$$

Harmonic oscillator: propagator

$$q(t) = \frac{1}{\sqrt{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t}), \quad [a, a^\dagger] = 1 .$$

Ground state $|0\rangle$: $a|0\rangle = 0$. The two-point function:

$$\langle 0|q(t)q(t')|0\rangle = \frac{1}{2\omega} e^{-i\omega(t-t')} .$$

Propagator: the time-ordered two-point function

$$\begin{aligned} \langle q(t)q(t')\rangle &= \theta(t-t')\langle 0|q(t)q(t')|0\rangle + \theta(t'-t)\langle 0|q(t')q(t)|0\rangle \\ &= \frac{1}{2\omega} e^{-i\omega|t-t'|} , \end{aligned}$$

$$\theta(t_1 - t_2) = \begin{cases} 1 & \text{if } t_1 > t_2 , \\ 0 & \text{if } t_1 < t_2 . \end{cases}$$

Harmonic oscillator: time derivatives of the propagator

$$\langle q(t)q(t')\rangle = \theta(t-t')\langle 0|q(t)q(t')|0\rangle + \theta(t'-t)\langle 0|q(t')q(t)|0\rangle\,.$$

$$\partial_t\theta(t-t') = \delta(t-t')\,, \qquad \partial_{t'}\theta(t-t') = -\delta(t-t')\,.$$

$$\begin{aligned}\partial_t\langle q(t)q'(t)\rangle &= \langle\dot{q}(t)q(t')\rangle \\ &\quad + \delta(t-t')\langle 0|q(t)q(t')|0\rangle - \delta(t-t')\langle 0|q(t')q(t)|0\rangle \\ &= \langle\dot{q}(t)q(t')\rangle + \delta(t-t')\langle 0|[q(t),q(t')]|0\rangle\,.\end{aligned}$$

$$\delta(t-t')[q(t),q(t')] = \delta(t-t')[q(t),q(t)] = 0\,.$$

$$\begin{aligned}\partial_t\langle q(t)q'(t)\rangle &= \langle\dot{q}(t)q(t')\rangle\,, \\ \partial_{t'}\langle q(t)q'(t)\rangle &= \langle q(t)\dot{q}(t')\rangle\,.\end{aligned}$$

Harmonic oscillator: time derivatives of the propagator

$$\partial_{t'}\langle q(t)q'(t)\rangle = \langle q(t)\dot{q}(t')\rangle \,.$$

In general

$$\partial_t\langle A(t)B(t')\rangle = \langle \dot{A}(t)B(t')\rangle + \delta(t-t')\langle 0|[A(t),B(t)]|0\rangle \,.$$

$$\begin{aligned}\partial_t\partial_{t'}\langle q(t)q(t')\rangle &= \partial_t\langle q(t)\dot{q}(t')\rangle \\ &= \langle \dot{q}(t)\dot{q}(t')\rangle + \delta(t-t')[q(t),\dot{q}(t')] \,.\end{aligned}$$

Hence

$$\boxed{\partial_t\partial_{t'}\langle q(t)q(t')\rangle = \langle \dot{q}(t)\dot{q}(t')\rangle + i\delta(t-t')} \,.$$

Difference in the propagators

$$\boxed{\partial_t \partial_{t'} \langle q(t) q(t') \rangle = \langle \dot{q}(t) \dot{q}(t') \rangle + i\delta(t - t')}\,.$$

The propagator in Lagrangian perturbation theory:

$$\dot{q}(t)\dot{q}(t') \mapsto \partial_t \partial_{t'} \langle q(t) q(t') \rangle\,.$$

The propagator in Hamiltonian perturbation theory:

$$\dot{q}(t)\dot{q}(t') \mapsto \langle \dot{q}(t) \dot{q}(t') \rangle\,,$$

i.e., $\langle \dot{q}(t) \dot{q}(t') \rangle_L = \langle \dot{q}(t) \dot{q}(t') \rangle_H + i\delta(t - t')$.

$$\boxed{\dot{q}_L(t) = \dot{q}(t) + \alpha(t)\,, \quad \langle \alpha(t) \alpha(t') \rangle = i\delta(t - t')\,.}$$

Perturbation theories (no time derivative)

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) + L_I(q)\,, \qquad -L_I(q) = \frac{\lambda}{3!} q^3\,.$$

The N-point function:

$$T\langle\Omega|q_H(t_1)q_H(t_2)\cdots q_H(t_N)|\Omega\rangle$$
$$= T\langle 0|q(t_1)q(t_2)\cdots q(t_N)\exp\left(-i\int[-L_I(q(\tau))]d\tau\right)|0\rangle_C\,,$$

$q(t)$: free operator; $|\Omega\rangle$: ground state; $|0\rangle$: perturbative ground state; C: "connected".

$$\exp\left(-i\int[-L_I(q(\tau))]d\tau\right) = \sum_{k=0}^{\infty}\frac{1}{n!}\left(-i\int[-L_I(q(\tau))]d\tau\right)^n\,.$$

Perturbation theories (no time derivative)

The two-point function at second order:

$$T\langle\Omega|q_H(t_1)q_H(t_2)|\Omega\rangle$$
$$= (-i)^2 \cdot \frac{\lambda^2}{2!(3!)^2} T\langle 0|q(t_1)q(t_2)\int d\tau_1 \int d\tau_2 [q(\tau_1)]^3[q(\tau_2)]^3|0\rangle_C \,.$$

The Wick Theorem:

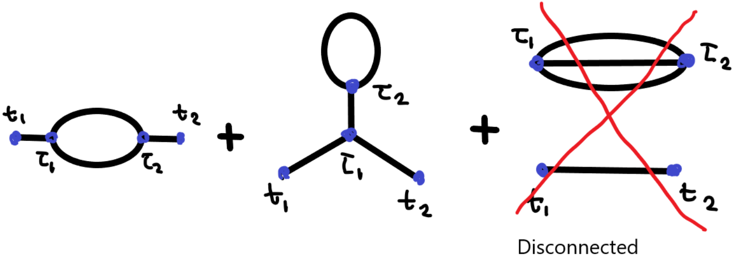
$$\begin{aligned} T\langle 0|q(t_1)q(t_2)q(t_3)q(t_4)|0\rangle = & \langle q(t_1)q(t_2)\rangle\langle q(t_3)q(t_4)\rangle \\ & + \langle q(t_1)q(t_3)\rangle\langle q(t_2)q(t_4)\rangle \\ & + \langle q(t_1)q(t_4)\rangle\langle q(t_2)q(t_3)\rangle \,. \end{aligned} \quad (1)$$

- The RHS: the sum of all possible parings of $q(t_1)$, $q(t_2)$, $q(t_3)$ and $q(t_4)$, which generalizes to the products of $q(t_1)$, $q(t_2)$,...,$q(t_{2N})$.

Perturbation theories (no time derivative)

$$
\begin{aligned}
&T\langle\Omega|q_H(t_1)q_H(t_2)|\Omega\rangle_{(2)}\\
&= -\frac{1}{2}\int d\tau_1 d\tau_2 \langle q(t_1)q(\tau_1)\rangle\langle q(t_2)q(\tau_2)\rangle(\langle q(\tau_1)q(\tau_2)\rangle)^2\\
&\quad -\frac{1}{2}\int d\tau_1 d\tau_2 \langle q(t_1)q(\tau_1)\rangle\langle q(t_2)q(\tau_1)\rangle\langle q(\tau_1)q(\tau_2)\rangle\langle[q(\tau_2)]^2\rangle\,.
\end{aligned}
$$

The graphical representation:



Perturbation theories (no time derivative)

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) + L_I(q)\,,$$

$L_I(q)$ contains no $\dot{q}$, e.g. $L_I(q) = -(\lambda/3!)q^3$.
Hamiltonian H?

$$\text{canonical conjugate momentum:} \quad p = \frac{\partial L}{\partial \dot{q}} = \dot{q}\,.$$

$$\begin{aligned} H &= p\dot{q} - L \\ &= p^2 - \frac{1}{2}(p^2 - \omega^2 q^2) - L_I(q) \\ &= \frac{1}{2}(p^2 + \omega^2 q^2) - L_I(q)\,. \end{aligned}$$

$$H_I(q) = -L_I(q)\,.$$

Perturbation theories (no time derivative)

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) + L_I(q)\,,$$
$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + H_I(q)\,,$$
$$p = \dot{q} \qquad H_I(q) = -L_I(q).$$

$$T\langle\Omega|q_H(t_1)q_H(t_2)\cdots q_H(t_N)|\Omega\rangle$$
$$= T\langle 0|q(t_1)q(t_2)\cdots q(t_N)\exp\left(-i\int[-L_I(q(\tau))]d\tau\right)|0\rangle_C$$
$$= T\langle 0|q(t_1)q(t_2)\cdots q(t_N)\exp\left(-i\int H_I(q(\tau))d\tau\right)|0\rangle_C$$

Perturbation theories (with time derivatives)

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) - \dot{q}F(q) - V(q)\,.$$

$$-L_I(q) = \dot{q}F(q) + V(q)\,.$$

$$\text{canonical conjugate momentum:} \quad p = \frac{\partial L}{\partial \dot{q}} = \dot{q} - F(q)\,.$$

$$\begin{aligned} H &= p\dot{q} - L \\ &= p[p + F(q)] - \frac{1}{2}[p + F(q)]^2 + \frac{1}{2}\omega^2 q^2 + [p + F(q)]F(q) + V(q) \\ &= \frac{1}{2}(p^2 + \omega^2 q^2) + pF(q) + V(q) + \frac{1}{2}[F(q)]^2\,. \end{aligned}$$

Perturbation theories (with time derivative)

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) + L_I(q, \dot{q})\,,$$
$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + H_I(q, p)\,,$$

where

$$-L_I(q, \dot{q}) = \dot{q}F(q) + V(q)\,,$$
$$H_I(q, p) = pF(q) + V(q) + \frac{1}{2}[F(q)]^2\,.$$

In perturbation theory, the quantum operators appearing in the interaction term are free operators. $\Rightarrow p = \dot{q}$

$$H_I(q, \dot{q}) = -L_I(q, \dot{q}) + \frac{1}{2}[F(q)]^2\,.$$

Perturbation theories (with time derivative)

$$
\begin{aligned}
&T\langle\Omega|q_H(t_1)q_H(t_2)|\Omega\rangle \\
&= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\{-L_I(q(\tau),\dot{q}_L(\tau))\}\right)|0\rangle_C \\
&= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau H_I(q(\tau),\dot{q}(\tau))\right)|0\rangle_C\,,
\end{aligned}
$$

$$
\begin{aligned}
-L_I(q,\dot{q}_L) &= \dot{q}_L F(q) + V(q)\,, \\
H_I(q,\dot{q}) &= -L_I(q,\dot{q}) + \frac{1}{2}[F(q)]^2\,.
\end{aligned}
$$

$$
\boxed{\dot{q}_L(t) = \dot{q}(t) + \alpha(t)\,,\quad \langle\alpha(t)\alpha(t')\rangle = i\delta(t-t')\,.}
$$

Perturbation theories (with time derivative)

With $q_L(\tau) = \dot{q}(\tau) + \alpha(\tau)$,

$$\exp\left(-i\int d\tau\{-L_I(q(\tau),\dot{q}_L(\tau))\}\right)$$
$$= \exp\left(-i\int d\tau[\dot{q}_L(\tau)F(q(\tau)) + V(q(\tau))]\right)$$
$$= \exp\left(-i\int d\tau[\dot{q}(\tau)F(q(\tau)) + V(q(\tau))]\right)$$
$$\times \exp\left(-i\int d\tau\alpha(\tau)F(q(\tau))\right) .$$

Perturbation theories (with time derivative)

$$
\begin{aligned}
&T\langle 0|\cdots \exp\left(-i\int d\tau\alpha(\tau)F(q(\tau))\right)|0\rangle_C \\
&= T\langle 0|\cdots \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\left(\int d\tau\,\alpha(\tau)F(q(\tau))\right)^N|0\rangle_C\,.
\end{aligned}
$$

$$
\begin{aligned}
&T\langle 0|\cdots\int d\tau\,\alpha(\tau)F(q(\tau))\cdots\int d\tau'\,\alpha(\tau')F(q(\tau'))|0\rangle \\
&= T\langle 0|\cdots\int d\tau\int d\tau' F(q(\tau))\,i\delta(\tau-\tau')\,F(q(\tau'))|0\rangle \\
&= iT\langle 0|\cdots\int d\tau[F(q(\tau))]^2|0\rangle\,.
\end{aligned}
$$

Perturbation theories (with time derivative)

$$T\langle 0|\cdots\sum_{n=0}^{\infty}\frac{(-i)^{2n}}{(2n)!}\left(\int d\tau\,\alpha(\tau)F(q(\tau))\right)^{2n}|0\rangle_C\,.$$

Number of ways to pair up $2n$ α's?

The number of ways to order $2n$ α's $=(2n)!$.

We pair 1st-2nd, 3rd-4th, 5th-6th, $\cdots$.

Over-counting:

- The ordering of pairs: $n!$.
- The ordering in the pair: 2^n.

The number of ways to pair up α's $=(2n)!/[2^n n!]$.

Perturbation theories (with time derivative)

$$
\begin{aligned}
&T\langle 0|\cdots\sum_{n=0}^{\infty}\frac{(-i)^{2n}}{(2n)!}\left(\int d\tau\,\alpha(\tau)F(q(\tau))\right)^{2n}|0\rangle_C\\
&=T\langle 0|\cdots\sum_{n=0}^{\infty}\frac{(-1)^n i^n}{2^n n!}\left(\int d\tau\,[F(q(\tau))]^2\right)^{n}|0\rangle_C\\
&=T\langle 0|\cdots\sum_{n=0}^{\infty}\left(-\frac{i}{2}\int d\tau\,[F(q(\tau))]^2\right)^{n}|0\rangle_C\\
&=T\langle 0|\cdots\cdots\exp\left(-i\int d\tau\,\frac{1}{2}[F(\tau)]^2\right)|0\rangle_C\,.
\end{aligned}
$$

Perturbation theories (with time derivative)

$$
\begin{aligned}
-L_I(q,\dot{q}) &= \dot{q}_L F(q) + V(q)\,, \\
H_I(q,\dot{q}) &= -L_I(q,\dot{q}) + \frac{1}{2}[F(q)]^2\,.
\end{aligned}
$$

$$
\begin{aligned}
&T\langle\Omega|q_H(t_1)q_H(t_2)|\Omega\rangle \\
&= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\,[-L_I(q(\tau),\dot{q}_L(\tau))]\right)|0\rangle_C \\
&\text{with}\quad \dot{q}_L(\tau) = \dot{q}(\tau) + \alpha(\tau)\,,\quad \langle\alpha(\tau)\alpha(\tau')\rangle = i\delta(\tau-\tau') \\
&= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\,\left\{-L_I(q(\tau),\dot{q}(\tau)) + \frac{1}{2}F(q(\tau))]^2\right\}\right) \\
&= T\langle 0|q(t_1)q(t_2)\exp\left[-i\int d\tau\,H_I(q(\tau),\dot{q}(\tau))\right]\,.
\end{aligned}
$$

Perturbation theories (with two time derivatives)

$$L(q,\dot{q}) = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) - \frac{1}{2}G(q)\dot{q}^2 - \dot{q}F(q) - V(q)\,,$$

$$-L_I(q,\dot{q}) = \frac{1}{2}G(q)\dot{q}^2 + \dot{q}F(q) + V(q)\,.$$

$$H(q,p) = \frac{1}{2}\frac{[p + F(q)]^2}{1 - G(q)} + \frac{1}{2}\omega^2 q^2 + V(q)\,.$$

With $p = \dot{q}$

$$H_I(q,\dot{q}) = -L_I(q,\dot{q}) + \frac{1}{2(1 - G(q))}(G(q)\dot{q} + F(q))^2\,.$$

Perturbation theories (with two time derivatives)

$$T\langle\Omega|q_H(t_1)q_H(t_2)|\Omega\rangle$$

$$= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\,[-L_I(q(\tau),\dot{q}_L(\tau))]\right)$$

$$\times\exp\left(-\frac{\delta(0)}{2}\int d\tau\,\log[1-G(q(\tau))]\right)|0\rangle_C\,. \leftarrow \text{ the Lee-Yang term}$$

$$(\delta(0)=\lim_{\tau\to\tau'}\delta(\tau-\tau'))\,.$$

$$(\text{with } \dot{q}_L(t)=\dot{q}(t)+\alpha(t)\,,\quad \langle\alpha(t)\alpha(t')\rangle = i\delta(t-t'))$$

$$= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\,[-L_I(q(\tau),\dot{q}(\tau))]\right)$$

$$\exp\left(-i\int d\tau\,\{[G(q(\tau))\dot{q}(\tau)+F(q(\tau))]\alpha(\tau)+G(q(\tau))[\alpha(\tau)]^2\}\right)$$

$$\times\exp\left(-\frac{\delta(0)}{2}\int d\tau\,\log[1-G(q(\tau))]\right)|0\rangle_C\,.$$

Perturbation theories (with two time derivatives)

$$
\begin{aligned}
& T\langle\Omega|q_H(t_1)q_H(t_2)|\Omega\rangle \\
&= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\,[-L_I(q(\tau),\dot{q}_L(\tau))]\right) \\
&\quad\times\exp\left(-i\int d\tau\,\frac{1}{2(1-G(q(\tau)))}[G(q(\tau))\dot{q}(\tau)+F(q(\tau))]^2\right)|0\rangle_C \\
&= T\langle 0|q(t_1)q(t_2)\exp\left(-i\int d\tau\,H_I(q(\tau),\dot{q}(\tau))]\right)|0\rangle_C\,,
\end{aligned}
$$

$$
\frac{1}{1-G(q)} = \sum_{n=1}^{\infty}[G(q)]^n\,,
$$

$$
\log[1-G(q)] = -\sum_{n=1}^{\infty}\frac{1}{n}[G(q)]^n\,.
$$

Concluding remarks

(A) Interaction terms are different in Hamiltonian and Lagrangian perturbation theories if interaction terms have time derivatives.

(B) The propagators for $\dot{q}(t)$ and $\dot{q}(t')$ are different in these perturbation theories:

$$\underbrace{\partial_t \partial_{t'} T\langle 0|q(t)q(t')|0\rangle}_{\text{Lagrangian}} = \underbrace{T\langle 0|\dot{q}(t)\dot{q}(t')|0\rangle}_{\text{Hamiltonian}} + i\delta(t-t')\,.$$

(A) is "corrected" by (B) and the two perturbation theories are equivalent as they should be.

- Generalization to more than one degree of freedom, to field theory?
- A more natural justification for the propagator in Lagrangian perturbation theory?