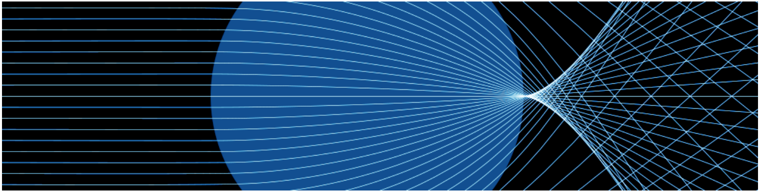
The Maxwell fisheye lens
and collapsing spheres of uniform density

Sam Dolan
University of Sheffield



IX Amazonian Workshop on Gravity and Analogue Models
18th June 2024.

Overview

1. Motivation: gravitational lensing
2. The Maxwell fisheye lens
3. Conformal symmetry

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4. The interior Schwarzschild solution
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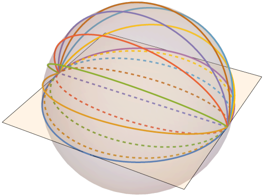
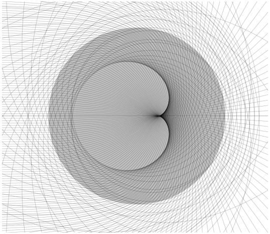
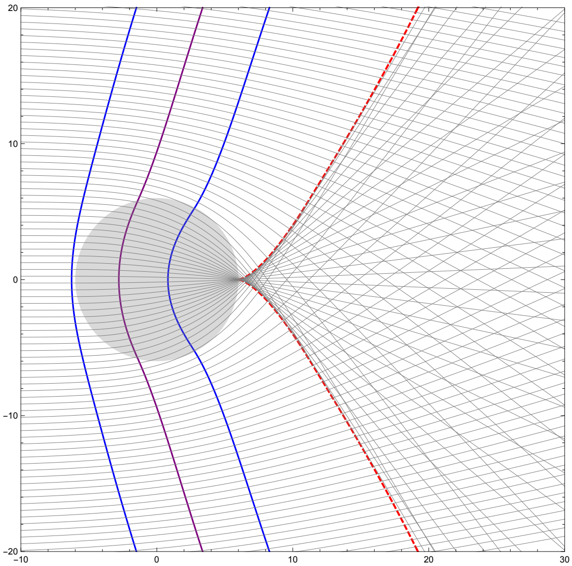
6. Gravitational collapse scenarios
7. Conclusions





Wavefronts passing through a compact body



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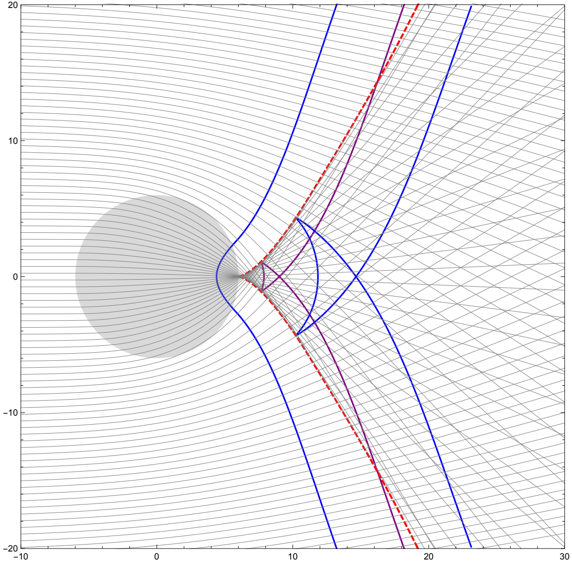

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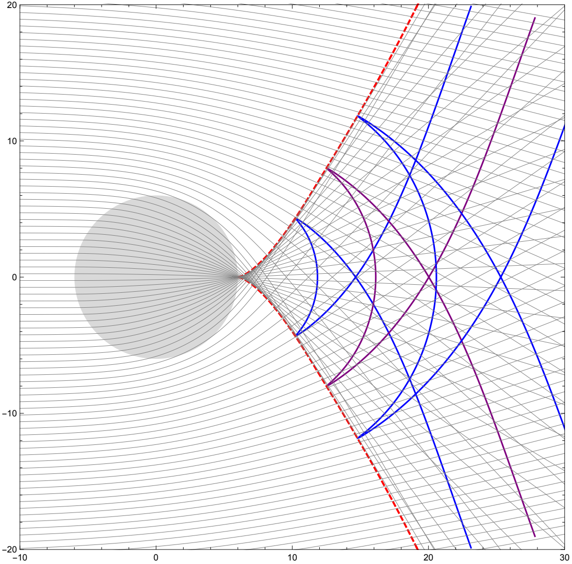
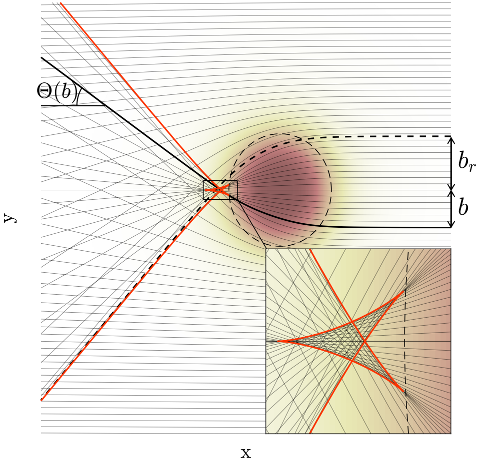


Waves passing over a submerged island



T. Torres, M. Lloyd, SD & S. Weinfurtner, Phys. Rev. Res. **4** (2022) 3, 033210.

Motivation

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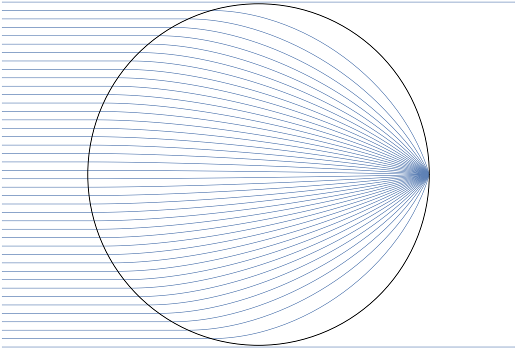

A. Prof. Shigeo Ohkubo: Consider **gradient-index lenses** in optics.

The Maxwell fisheye lens

- In 1853, a curious problem appeared in the *Dublin and Cambridge Mathematical Journal* (Problem 3, volume VIII, p188).
- The reader was challenged to find an optical refractive medium such that all the rays proceeding from **any** point in the medium will meet again accurately at another point, and such that the path of every ray in the medium is a segment of a circle.

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- The reader was challenged to find an optical refractive medium such that all the rays proceeding from **any** point in the medium will meet again accurately at another point, and such that the path of every ray in the medium is a segment of a circle.
- In the 1854 solution, the anonymous question-setter remarked that "The possibility of the existence of a medium of this kind possessing remarkable optical properties, was suggested by the contemplation of the structure of the crystalline lens in fish".
- Eleven years later, the solution appeared in *The Scientific Papers of James Clerk Maxwell.*

The Maxwell fisheye lens

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$$n(r) = \frac{2}{1 + r^2/\mathcal{R}^2}.$$

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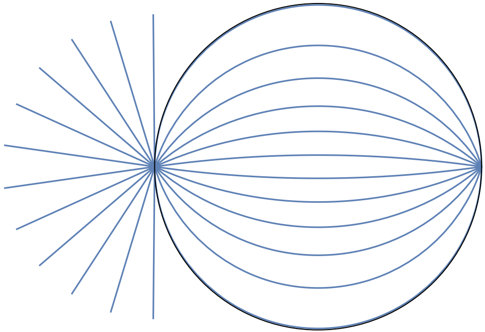
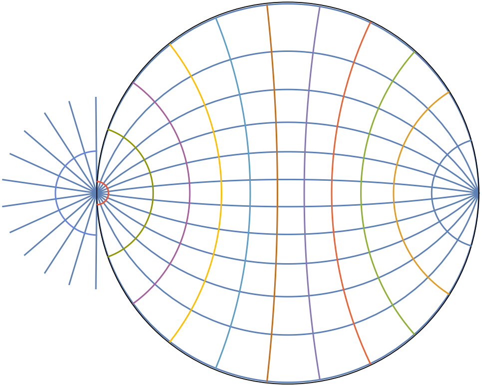
- Rays starting on the rim meet again on the opposite side.



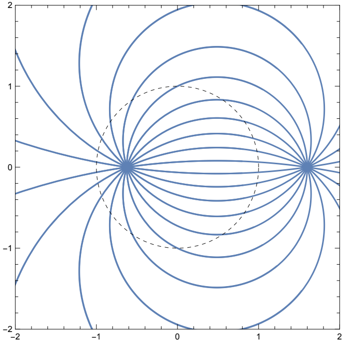
The Maxwell fisheye lens

- The wavefronts are orthogonal to the rays:



The extended Maxwell fisheye lens

- In the **extended** lens, rays emanating from any point $r = r_0$ are focussed at a conjugate point $r_1 = -\mathcal{R}^2/r_0$.



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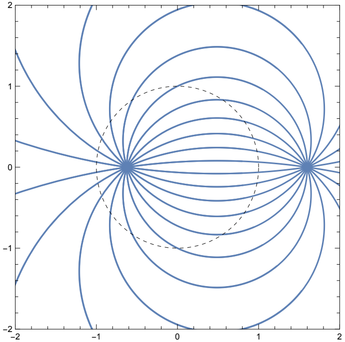

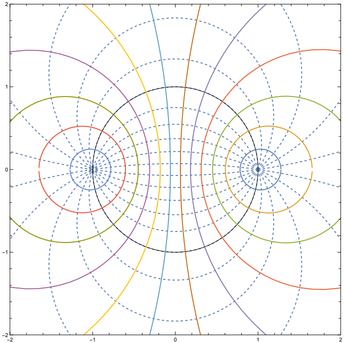

- The rays and wavefronts in an extended fisheye lens form **Apollonian circles**.

The extended Maxwell fisheye lens

- Is there a natural geometrical re-imagining?

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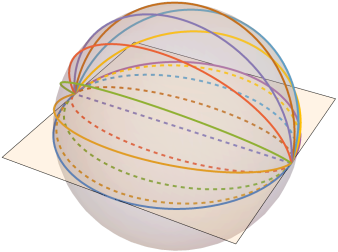

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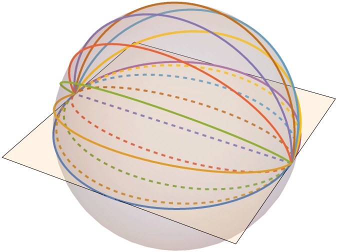

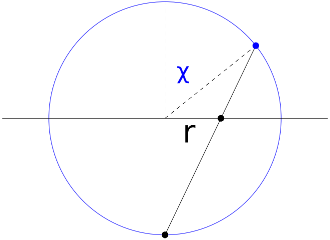

- Rays in the lens $\Leftrightarrow$ Null geodesics on a sphere

$$\boxed{r = \mathcal{R} \tan(\chi/2)}$$

Rays in a lens $\Leftrightarrow$ Null geodesics on a curved space

- Action principle: Fermat's principle of least time:

$$S_{\text{Fermat}} = \int_{t_A}^{t_B} dt = \frac{1}{c} \int_{x_A}^{x_B} n(x) d\ell,$$

where $d\ell = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} = \sqrt{\delta_{ij} dx^i dx^j}$.

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- Here $n(x)d\ell$ is the element of path length on a Riemannian space

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- The rays in the lens map to **null geodesics** of a **spacetime** with line element

$$ds^2 = -dt^2 + d\Sigma^2.$$

Rays in a lens $\Leftrightarrow$ Null geodesics on a curved space

- For Maxwell's fisheye lens,

$$d\Sigma^2 = \left(\frac{2}{1 + r^2/\mathcal{R}^2}\right)^2 \left(dr^2 + r^2 d\Omega_n^2\right)$$

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- This is the line element of a $(n+1)$-hypersphere.

Conformally-related spacetimes

- In the remainder of this talk, we will consider 4D spacetimes that are **conformal** to a hypersphere, with line element

$$ds^2 = \hat{\Omega}^2(x) \left(-dt^2 + d\Sigma^2\right), \quad d\Sigma^2 = \mathcal{R}^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right),$$

where $\hat{\Omega}(x) > 0$ everywhere, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

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- i.e. "Spacetimes conformal to the Maxwell fisheye lens".
- Conformally-related spacetimes share the same null geodesics.
- The conformal factor $\hat{\Omega}(x)$ can be a function of space **and time** (i.e. dynamics).

Conformal symmetry: key results

Consider two spacetimes related by a conformal factor:

$$S : \left(\mathcal{M}, g_{\mu\nu} = \hat{\Omega}^2(x)\tilde{g}_{\mu\nu}\right) \quad \text{and} \quad \tilde{S} : (\mathcal{M}, \tilde{g}_{\mu\nu})$$

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5. **Gravitational fields**. The Weyl tensor satisfies $\tilde{C}^{\mu}{}_{\nu\sigma\lambda} = C^{\mu}{}_{\nu\sigma\lambda}$.

Physical spacetimes

We now shift our attention to 'physical' spacetimes that:

- satisfy the **Einstein field equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

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Q. Are any 'physical' spacetimes conformal to Maxwell's fisheye lens?

The interior Schwarzschild solution

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- In Schwarzschild coordinates $\{t, r, \theta, \phi\}$,

$$ds^2 = -A(r)dt^2 + B^{-1}(r)dr^2 + r^2 d\Omega^2,$$

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- **Buchdahl bound:** the central pressure $P(0)$ diverges as $R \to 9M/4$.

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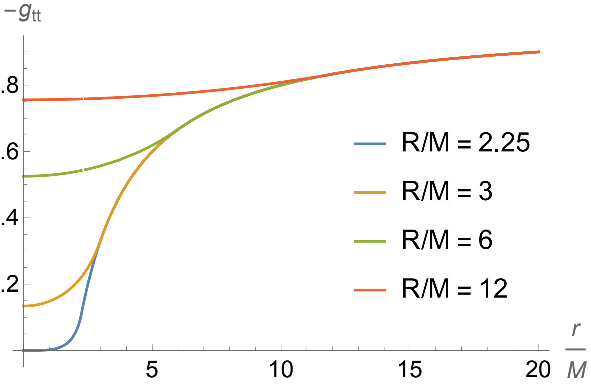

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- The interior solution can also be written in **isotropic coordinates** $\{t, \rho, \theta, \phi\}$ (Wyman 1946) as

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$$\mathbf{n}(\rho) = \frac{(1 + M/2\mathrm{a})^4}{2(1 - M/\mathrm{a})} \frac{2}{1 + \rho^2/\mathcal{R}^2}.$$

- Here a is the **isotropic radius** of the star:

$$R = \mathrm{a}\left(1 + \frac{M}{2\mathrm{a}}\right)^2 \quad \Leftrightarrow \quad \mathrm{a} = \frac{R}{2}\left(1 - M/R + \sqrt{1 - 2M/R}\right).$$

The interior Schwarzschild solution

- Now making the coordinate transformation $\rho = \mathcal{R} \tan(\chi/2)$,

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$$\mathcal{R} \equiv \sqrt{\frac{\mathrm{a}^3}{M} \frac{(1 - M/\mathrm{a})}{(1 - M/4\mathrm{a})}} \qquad \widehat{\mathcal{R}} = \frac{(1 + M/2\mathrm{a})^4}{2(1 - M/\mathrm{a})} \mathcal{R}.$$

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- The centre of the star is at $\chi = 0$, and its surface at $\chi = \chi_0$,

$$\chi_0 = 2\arctan\left(\sqrt{\frac{1-M/4\mathrm{a}}{\mathrm{a}/M-1}}\right).$$

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For a constant-density star embedded in Schwarzschild spacetime,

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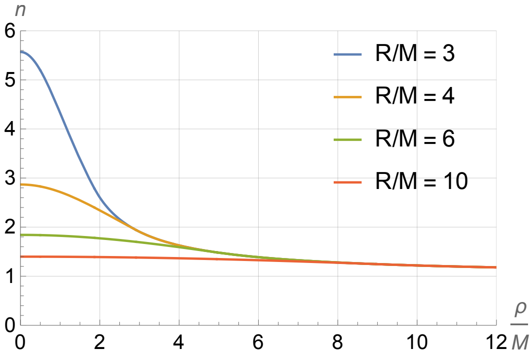

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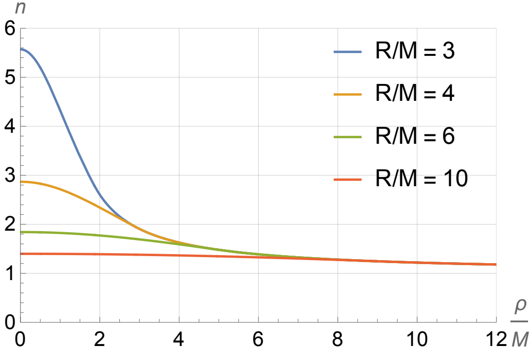

See also: W. Xiao and H. Chen, Optics Express 31, 11490 (2023).

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- **Q.** Are these two well-known spacetimes part of a **family**?

The Nariai-Tomita solution (1968)

H. Nariai & K. Tomita, Progress of Theoretical Physics **40**, 679 (1968).

$$ds^2 = e^{2\nu(\tau,\rho)} \left\{ -d\tau^2 + n^2(\tau,\rho) \left(d\rho^2 + \rho^2 d\Omega^2 \right) \right\}$$

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- **Q**. What is the physical meaning of the free functions $a(\tau)$, $b(\tau)$ and $\beta(\tau)$?

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- $\mathcal{E}_0$ is the **specific energy** and α_0 is the **proper acceleration** of the surface’s trajectory in the Schwarzschild spacetime.

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- In such cases, since

$$\mathcal{R}^2 \equiv \mathrm{a}_0^2 \frac{(1-\beta(\tau))}{\beta(\tau)}$$

then the proportion of the Maxwell fisheye lens encompassed by the interior is **also constant**.

Constant acceleration α_0 and uniform density

- By applying the standard coordinate transformation $\rho = \mathcal{R} \tan(\chi/2)$, the interior geometry takes the form

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- The interior Schwarzschild ($\alpha_0 > 0$) and Oppenheimer-Snyder collapse ($\alpha_0 = 0$) are **special cases** of the above result.

Asymptotic collapse

- An interesting special case is a uniform sphere which starts at $r_0 = R$ and whose (constant) proper acceleration α_0 is only just insufficient to prevent collapse: $\alpha_0 = (1-\epsilon)\frac{M}{R^2\sqrt{f(R)}}$, $\epsilon \ll 1$.

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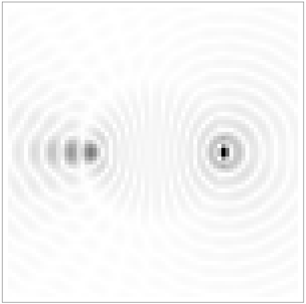
- In this scenario, we can obtain many results for fields and waves in closed form.



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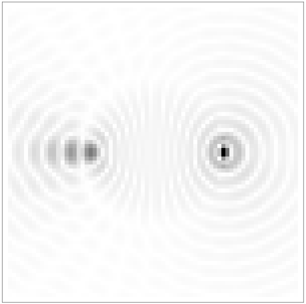

- **Warning:** None of the uniform density star models considered here satisfy realistic Equations of State, $\hat{\mu}(P)$.

Conclusions

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- Rays in the lens map to null geodesics on a (hyper)sphere.
- Several well-known solutions to the Einstein equations are *conformal* to hyperspherical geometries:
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- We have shown here that a wider class exists: Nariai-Tomita stars embedded in the Schwarzschild geometry.
- If the star's surface has a constant proper acceleration, then these geometries will focus null geodesics exactly like a fixed portion of a Maxwell fisheye lens.
- Using conformal symmetry, many results for fields and rays can be obtained in closed form for (simplified) collapse scenarios.