Dynamical ℓ-Boson Stars and relatives

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ℓ-Boson Stars

ℓ -BS are solutions to the static, spherically symmetric Einstein Klein-Gordon system for a collection of an arbitrary odd number N of complex scalar fields with no self-interactions.

Each non-interacting scalar field Φ, $i = 1,...,N$, of mass μ, is excited in an appropriate way consistent with the spherical symmetry of the spacetime.

The stress energy-momentum tensor associated with such a collection of scalar fields is given by $T_{\alpha\beta} = \sum_{i=1}^{N} T^{(i)}_{\alpha\beta}$,

$$T^{(i)}_{\alpha\beta} = (\nabla_\alpha\Phi_i\,\nabla_\beta\Phi_i^* + \nabla_\beta\Phi_i\,\nabla_\alpha\Phi_i^*) + g_{\alpha\beta}\left(\nabla_\sigma\Phi_i\nabla^\sigma\Phi_i^* + \frac{1}{2}\mu^2|\Phi_i|^2\right) .$$

Where * denotes the complex conjugate, and ∇ is the covariant derivative with respect to the spacetime metric g.

The conservation of the stress energy tensor implies each field must obey the Klein-Gordon Equation

$$(\nabla_\mu\nabla^\mu - \mu^2)\Phi_i = 0.$$

The model

We assume that each field has a harmonic time dependence, of the form

$$\Phi_{\ell m}(t,r,\vartheta,\varphi) = e^{-i\omega t}\psi_\ell(r) Y_{\ell,m}(\vartheta,\varphi) \; ,$$

Whereas the spacetime element of line is

$$ds^2 = -\alpha^2 dt^2 + \gamma^2 dr^2 + r^2 d\Omega^2,$$

where $\gamma^2 := \dfrac{1}{1-\frac{2M}{r}},$

The Einstein equations

$$M' = \frac{\kappa_\ell r^2}{2}\left[\frac{\psi_\ell'^2}{\gamma^2} + \left(\mu^2 + \frac{\omega^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right],$$

$$\frac{(\alpha\gamma)'}{\alpha\gamma^3} = \kappa_\ell r\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2\right],$$

The Klein Gordon equation

$$\frac{1}{r^2\alpha\gamma}\left(\frac{r^2\alpha}{\gamma}\psi_\ell'\right)' = \left(\mu^2 - \frac{\omega^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell.$$

In order to preserve the spherical symmetry of the configuration, all the fields must have the same amplitude.

Equilibrium configurations

For a real frequency ω, this system provides a nonlinear eigenvalue problem for the metric functions and the scalar field amplitude.

For the case of $N = 1$, i.e. $\ell = 0$, these equations reduce to the ones describing static mini - boson stars.

The nonlinear system has to be completed by giving boundary conditions. We assume the scalar field vanishes at infinity, thus the spacetime is asymptotically flat.

The solution of the system can be found by means of a shooting algorithm using ω as a shooting parameter.

Scaling, mass and size

The system is invariant under transformations of the form

$$\mu \mapsto \lambda\mu, \quad \omega \mapsto \lambda\omega, \quad r \mapsto \lambda^{-1} r, \quad u_\ell \mapsto \lambda^\ell u_\ell, \qquad \text{where} \qquad u_\ell := \psi_\ell / r^\ell$$

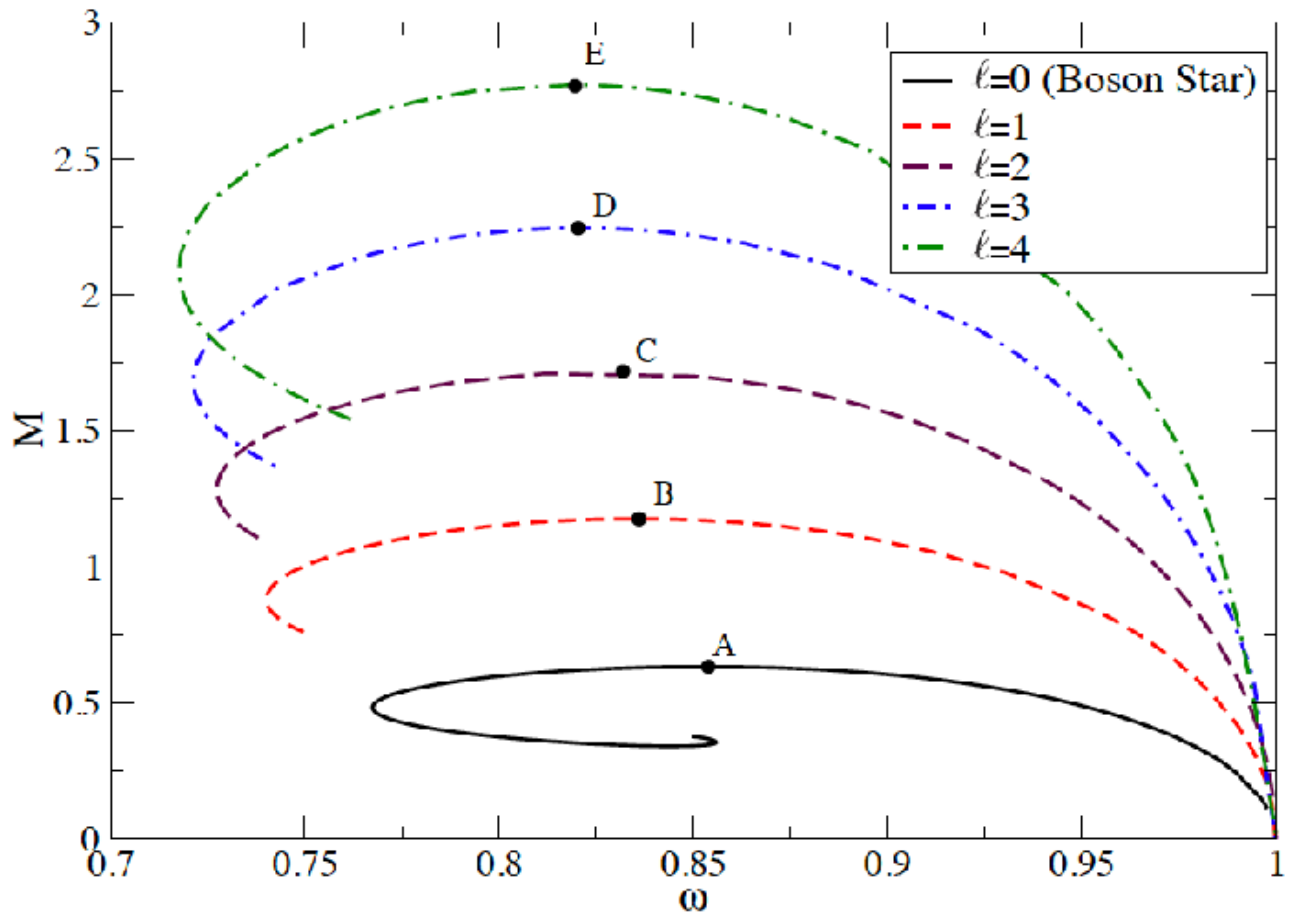
We characterize the total mass of an ℓ-boson star in terms of the asymptotic value of the Misner-Sharp mass function, which is approximated by evaluating the metric coefficient $\gamma(r)$ at the last grid point of the computational domain

$$M \approx \frac{r_{\max}}{2}\left[1 - \frac{1}{\gamma^2(r_{\max})}\right].$$

ℓ-Boson Stars extend to infinity and thus do not posses a surface at a finite radius, one can however, define an effective radius, R(99), as the areal radius of the object which contains 99% of the total mass.

For a given angular momentum number ℓ, the equilibrium configurations are labeled by a continuous parameter corresponding to the field amplitude.

Equilibrium configurations



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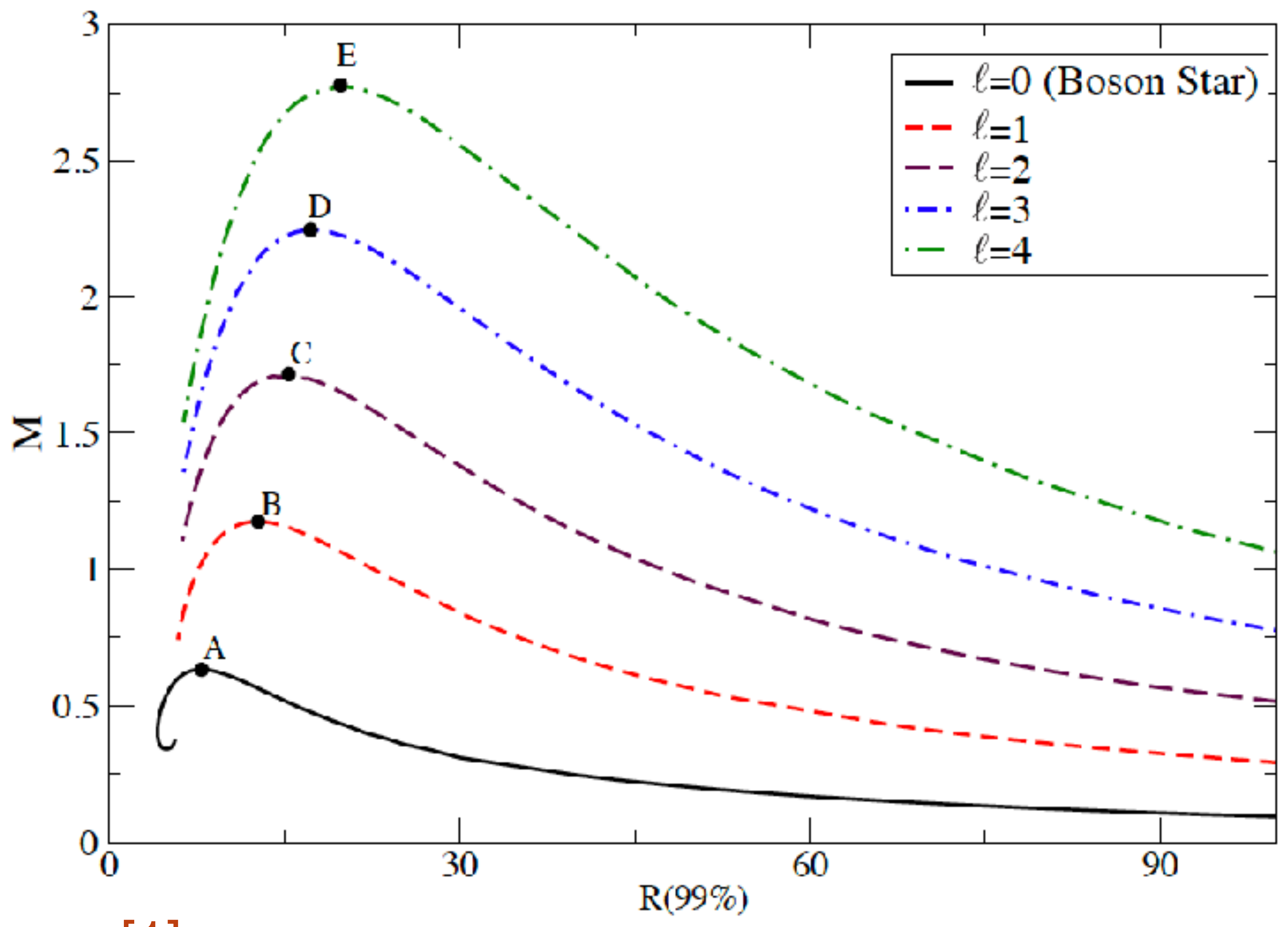

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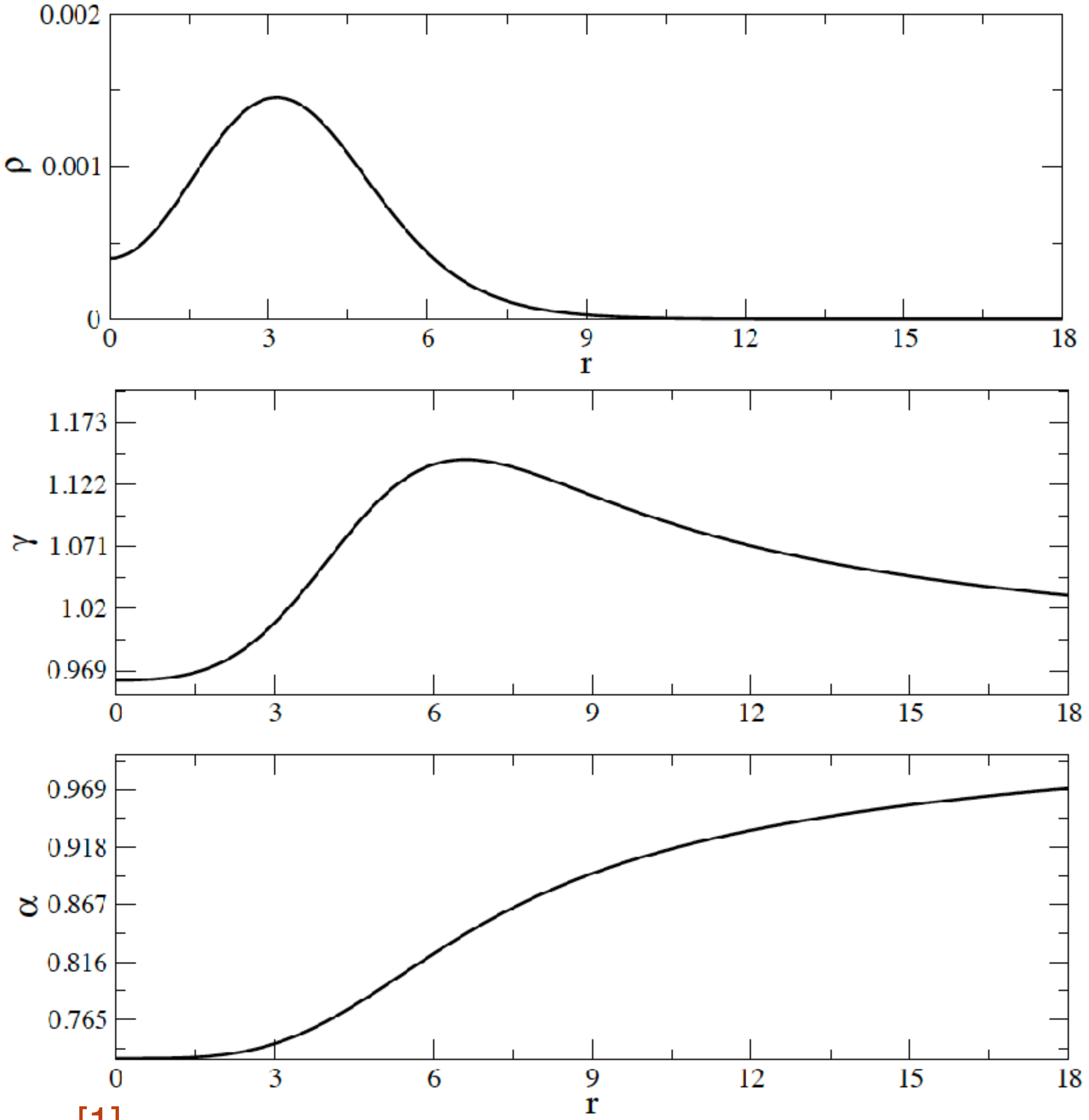
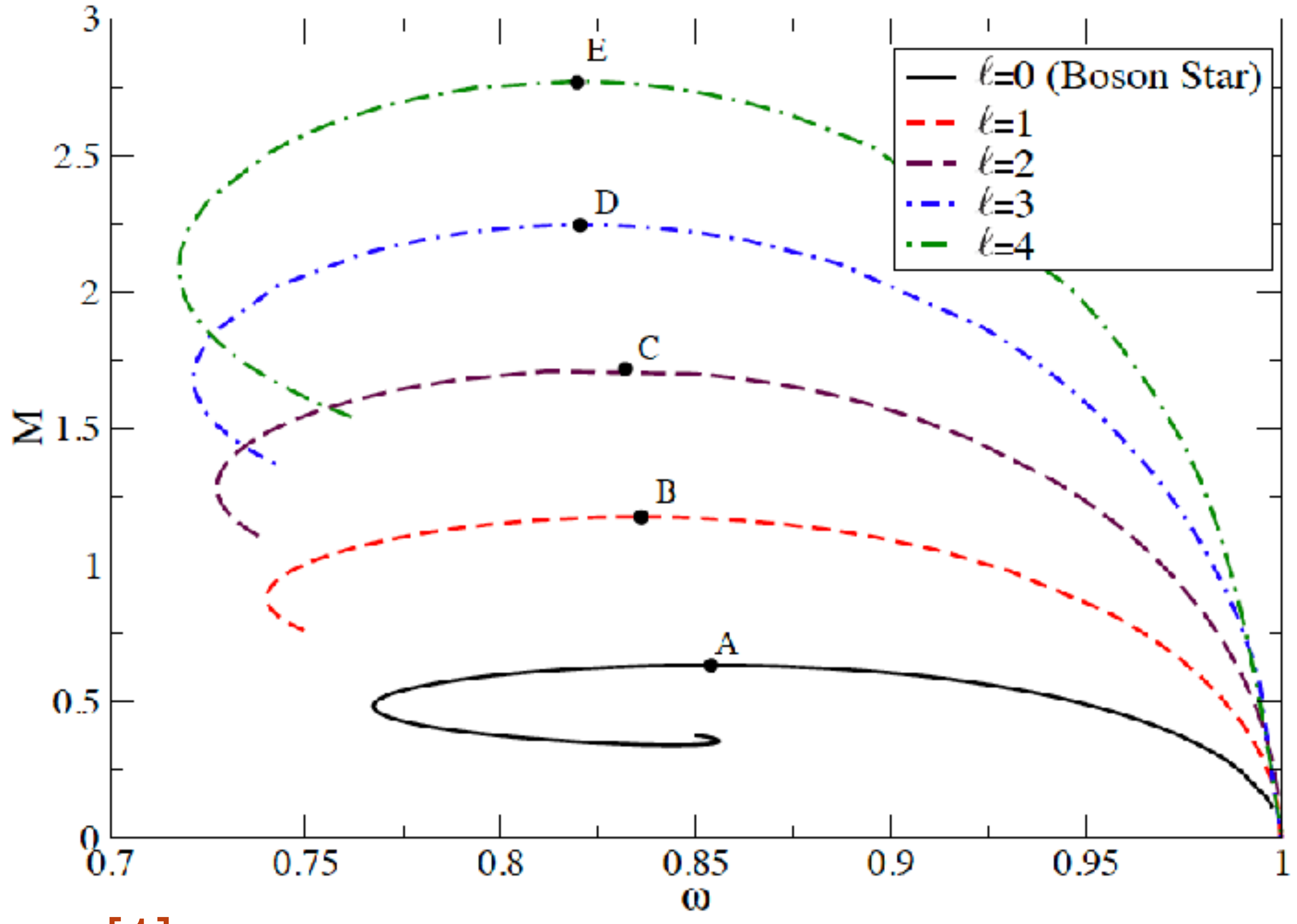
For a given value of ℓ, the mass M of the equilibrium configurations as a function of ω. As ℓ increases the configurations become more compact.

Equilibrium configurations



[1]
Energy density, and metric coefficients.



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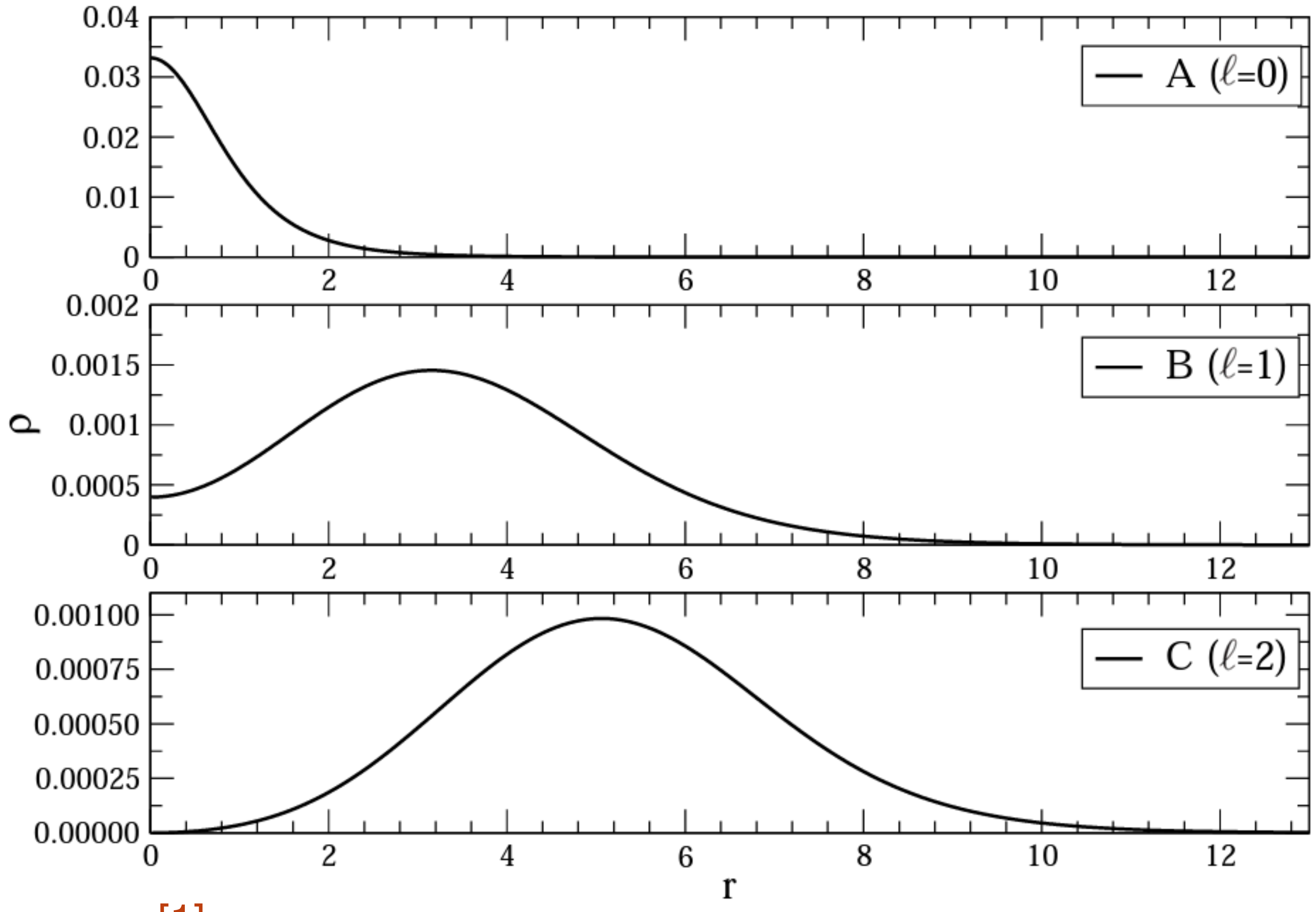
Configuration	M	$R(99\%)$	ω	$M/R(99\%)$
A $(\ell = 0)$	0.63	7.89	0.854	0.08
B $(\ell = 1)$	1.18	12.75	0.836	0.09
C $(\ell = 2)$	1.72	15.35	0.832	0.11
D $(\ell = 3)$	2.25	17.22	0.820	0.13
E $(\ell = 4)$	2.78	19.80	0.819	0.14

Compactness

Energy density



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For $\ell = 0$ the maximum value of the energy density (measured by Eulerian observers) is at the origin, whereas for $\ell > 0$ the structure of the stars is like a shell

Stability with respect to radial perturbations

For each value of ℓ, the configuration of maximum mass separates the parameter space into stable and unstable regions.

We perform non-linear evolutions of the coupled EKG systems, to determine the stability properties of the equilibrium states

Stable configurations, when perturbed, oscillate around the unperturbed solution and very slowly return to a stationary configuration.

Unstable configurations, in contrast, can have three different final states:

a) Collapse to a black hole,

b) Migration to the stable branch, or

c) Dissipation to infinity.

Perturbing the equilibrium states

In order to solve the field equations, we consider a spherically symmetric spacetime with a line element given by

$$ds^2 = -\alpha^2 dt^2 + \psi^4 \left(A dr^2 + r^2 B d\Omega^2 \right) \ ,$$

and the fields $\quad \Phi_{\ell m}(t,r,\vartheta,\varphi) = \phi_\ell(t,r) Y^{\ell m}(\vartheta,\varphi),$

The KG equation takes the form

$$\partial_t \Pi \ = \ \frac{\alpha}{A\psi^4}\left[\partial_r \chi + \chi \left(\frac{2}{r} - \frac{\partial_r A}{2A} + \frac{\partial_r B}{B} + 2\,\partial_r \psi \right)\right] \ + \ \frac{\chi\,\partial_r \alpha}{A\psi^4} + \alpha K \Pi - \alpha \left(\mu^2 + \frac{\ell(\ell+1)}{r^2 B \psi^4} \right) \ ,$$

$$\begin{aligned} \partial_t \phi \ &= \ \alpha \Pi \ , \\ \partial_t \chi \ &= \ \alpha \partial_r \Pi + \Pi \partial_r \alpha \ , \end{aligned} \qquad \chi := \partial_r \phi \ , \qquad \Pi := \frac{\partial_t \phi}{\alpha} \ .$$

We use for our dynamical simulations a spherically symmetric version of the Baumgarte-Shapiro-Shibata-Nakamura formulation with matter sources given by

$$\begin{aligned} \rho_E \ &= \ n^\mu n^\nu T_{\mu\nu} \\ &= \ \frac{1}{2}\left[|\Pi|^2 + \frac{|\chi|^2}{A\psi^4} + \left(\mu^2 + \frac{\ell(\ell+1)}{r^2} \right) |\phi|^2 \right], \\ P_r \ &= \ -n^\mu T_{r\mu} = -\frac{1}{2}\left(\chi \Pi^* + \Pi \chi^* \right), \end{aligned}$$

$$\begin{aligned} S^r_r \ &= \ \frac{1}{2}\left[|\Pi|^2 + \frac{|\chi|^2}{A\psi^4} - \left(\mu^2 + \frac{\ell(\ell+1)}{r^2} \right) |\phi|^2 \right], \\ S^\theta_\theta \ &= \ \frac{1}{2}\left[|\Pi|^2 - \frac{|\chi|^2}{A\psi^4} - \mu^2 |\phi|^2 \right], \end{aligned}$$

Initial data perturbations

In order to find the perturbed initial data we choose a value of ℓ and solve for the unperturbed configuration.

Having found the metric functions, the amplitude of the scalar field and the frequency we add small perturbations to the field and its time derivative and solve again the Hamiltonian constraint to find the modified metric radial function

We consider perturbations in the field of the form

$$\phi_R = \varphi_0 + \delta\varphi_R \;, \qquad \phi_I = \delta\varphi_I \;,$$
$$\Pi_R = \delta\Pi_R \;, \qquad \Pi_I = (\Pi_I)_0 + \delta\Pi_I \;,$$

Useful quantities are density of energy and density of bosons

$$\begin{aligned} \rho_E &= \frac{1}{2}\left[|\Pi|^2 + \frac{|\chi|^2}{A\psi^4} + \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right)|\phi|^2\right], \\ \rho_B &= -n^\mu J_\mu = \phi_R\Pi_I - \phi_I\Pi_R \;, \end{aligned}$$

where the conserved current is

$$J^\mu = -\frac{1}{2}\,\mathrm{Im}\left(\phi^*\nabla^\mu\phi - \phi\,\nabla^\mu\phi^*\right) \;,$$

Initial data perturbations

We consider three different types of perturbations

Type I. Internal perturbations such that the boson density changes

$$\delta\varphi_R \neq 0 \text{ and } \delta\Pi_I = 0.$$

Type II Internal perturbations such that the boson density is conserved to linear order and can increase or decrease the total mass of the star.

$$\delta\Pi_I = -(\omega/\alpha_0)\, \delta\varphi_R.$$

Type III. External perturbations such that the boson density is conserved to linear order, but always increases the mass

$$\delta\Pi_I = \pm(\omega/\alpha_0)\, \delta\varphi_R,$$

In all the simulations we consider perturbations of the form

$$\delta\varphi_R(r) = \epsilon \; \exp\left[-(r - r_0)/\sigma^2\right] \; ,$$

Diagnostics

The total number of particles

$$N_B \;:=\; \int \rho_B \gamma^{1/2} dr d\theta d\varphi$$

The binding energy is a measure of the difference between the total mass energy of the system, given by the ADM mass *M*, and the rest mass of the bosons, which can be simply defined as μN_B, with μ the mass of the scalar field

$$U := M - \mu N_B \; .$$

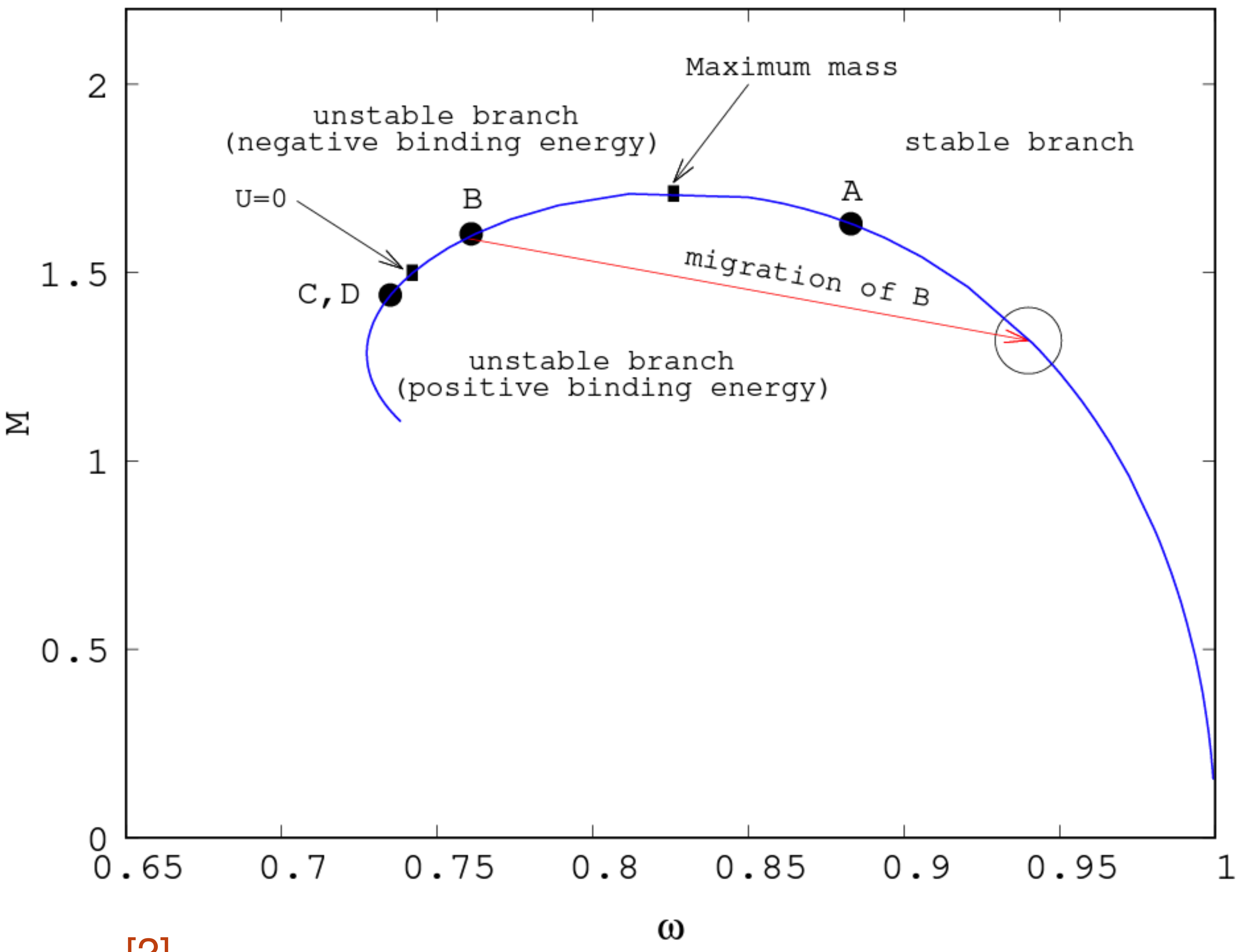
If the binding energy is negative, we should have a bound gravitational system, while if it is positive the system is unbound.

The formation of a black hole is monitored via the apparent horizon with mass

$$M_{\rm H} = \sqrt{\frac{A_{\rm H}}{16\pi}}$$

Representative cases ℓ = 2

Configuration A corresponds to a perturbation of a stable solution, configuration B to a perturbation of an unstable but bound solution, while configurations C and D correspond to different perturbations of the same unstable and unbound solution.



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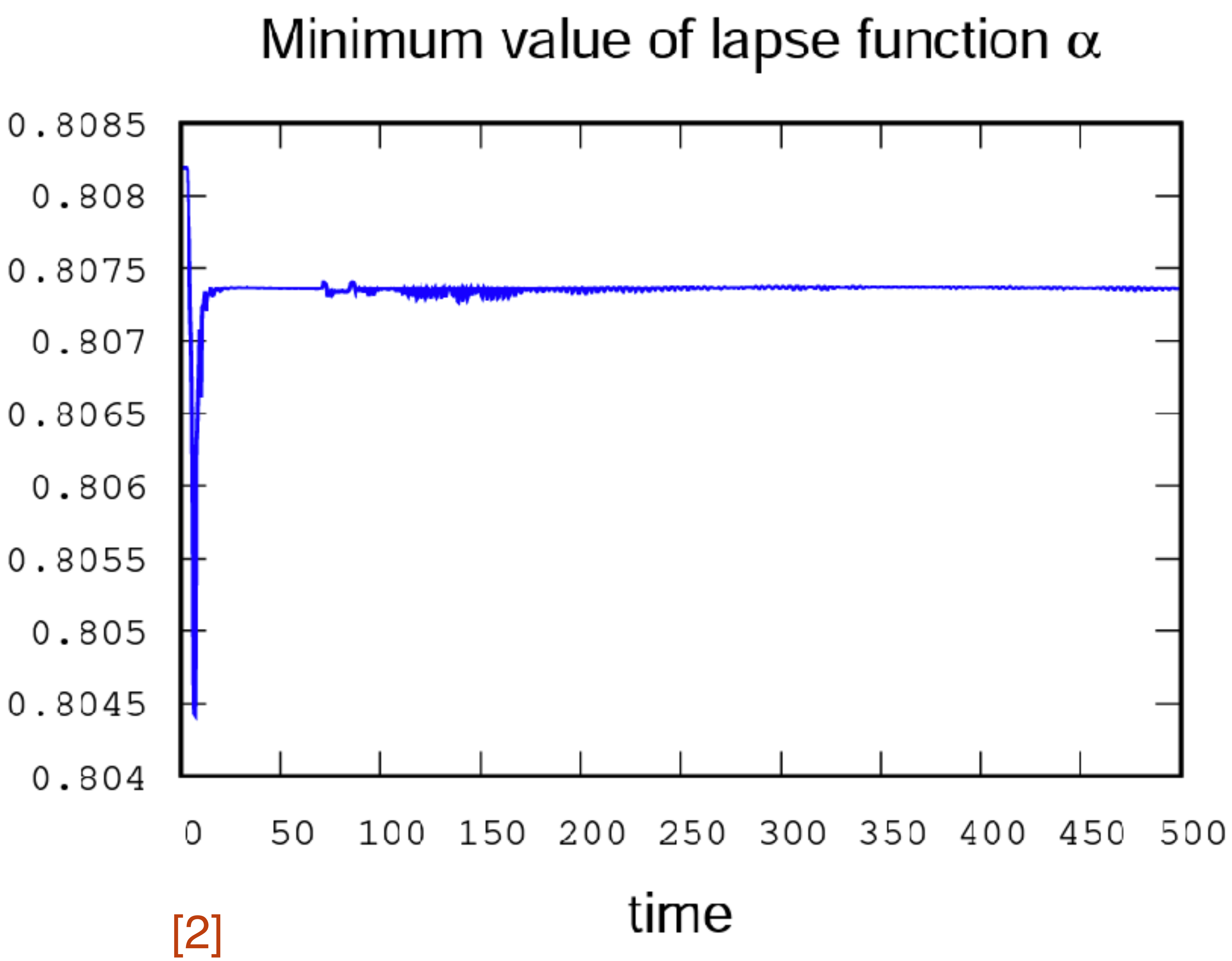
Stable configuration A



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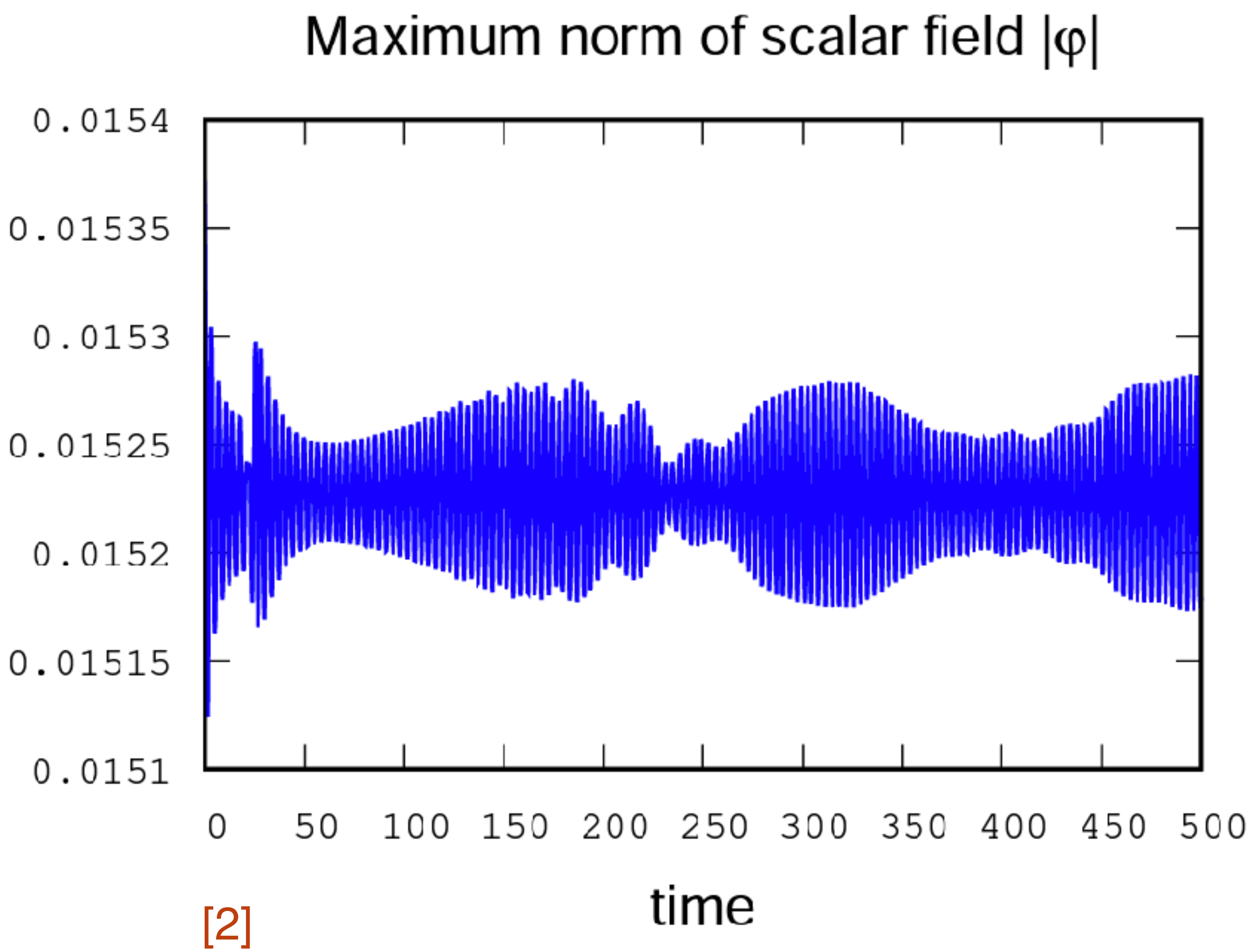

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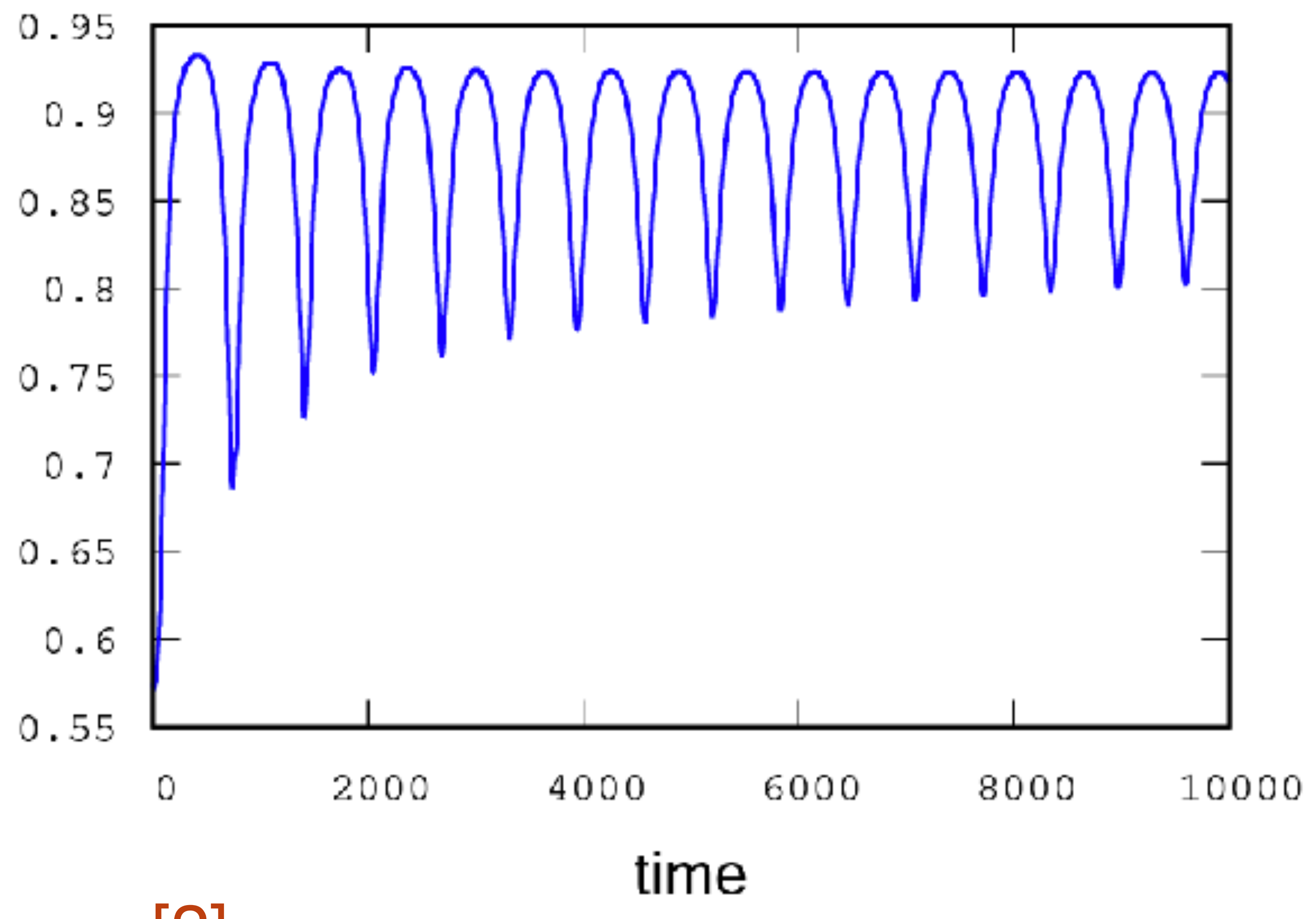
Unstable configuration B (Migration)

Minimum value of lapse function α



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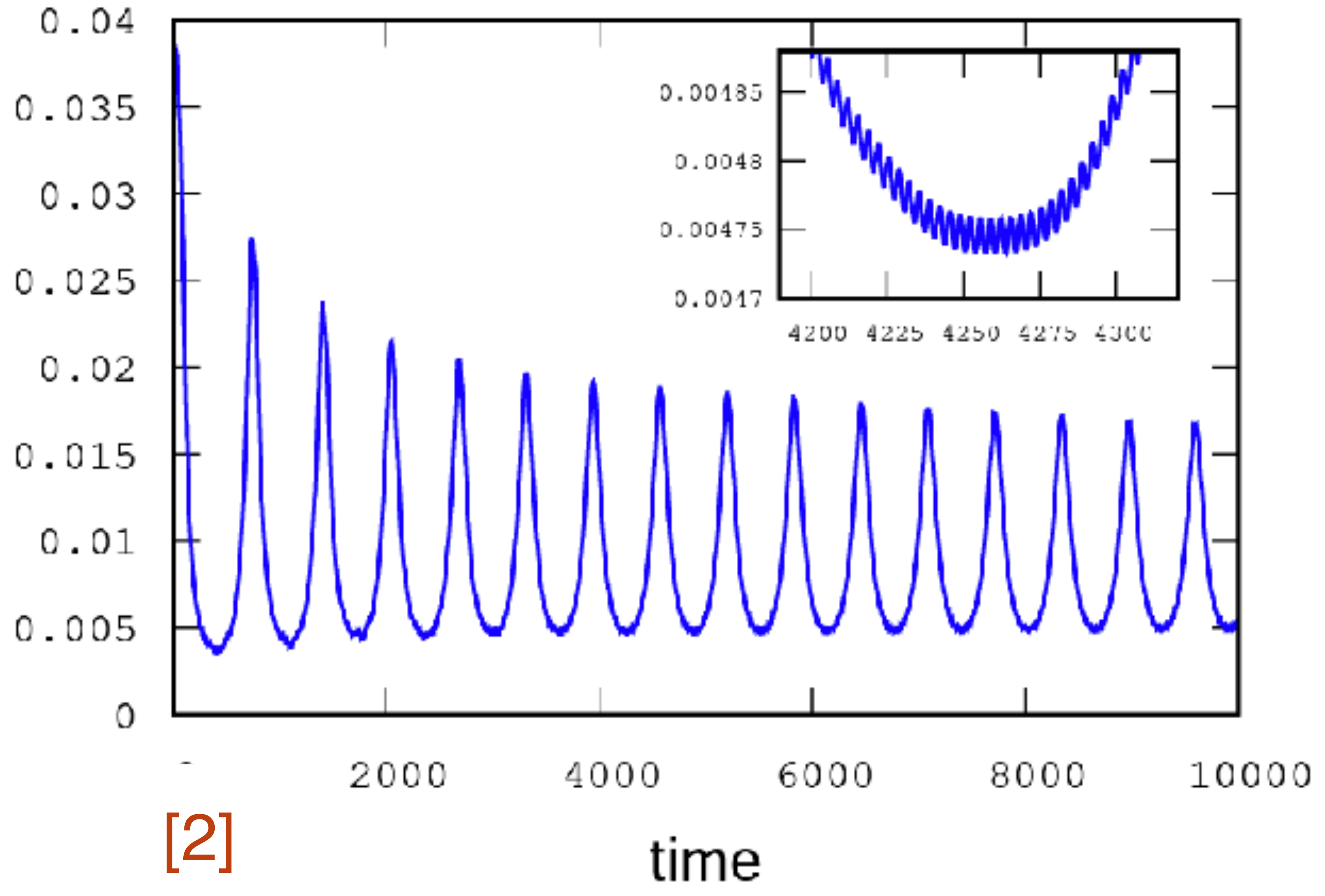
Maximum norm of scalar field |φ|



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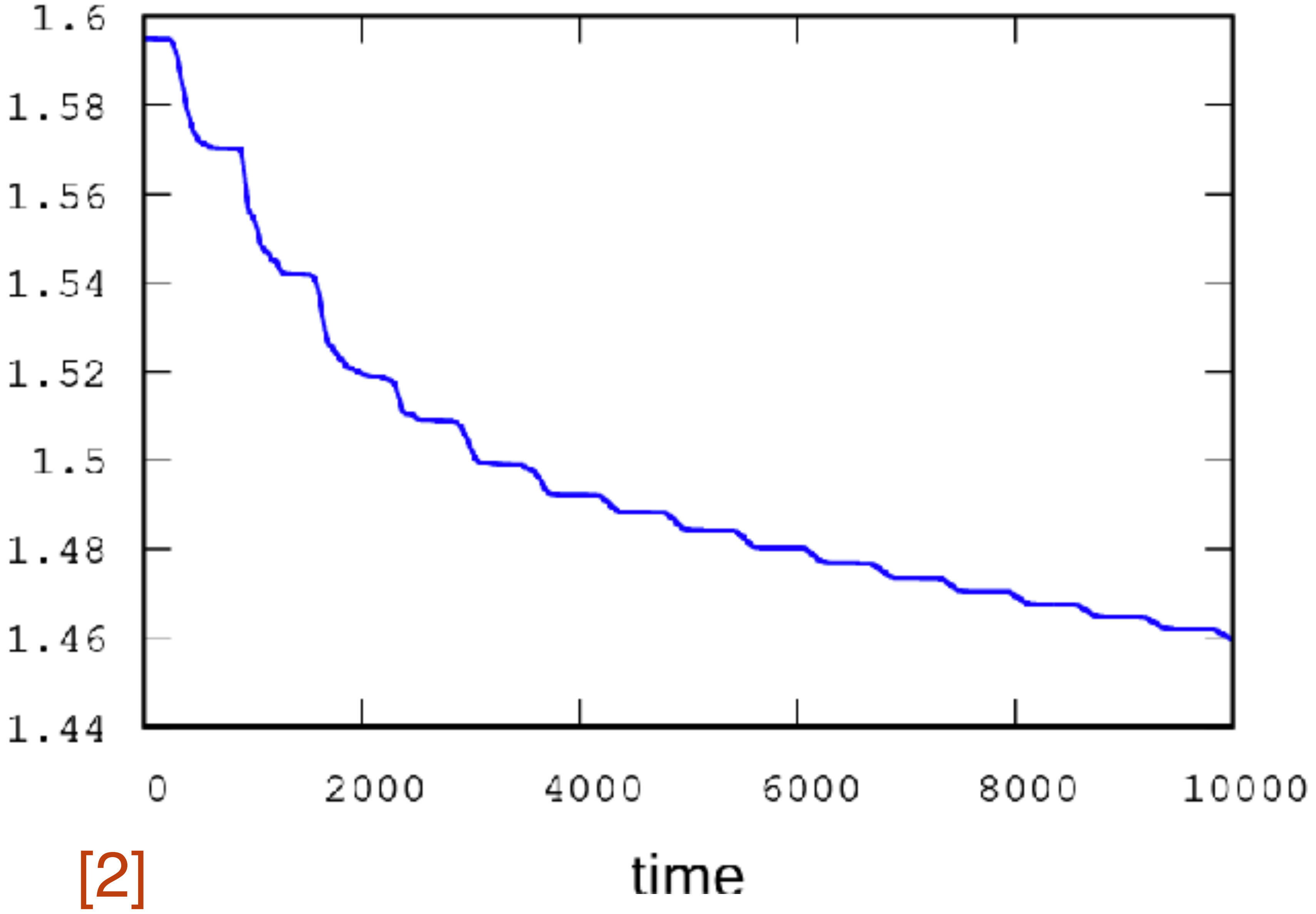
Mass M at boundary



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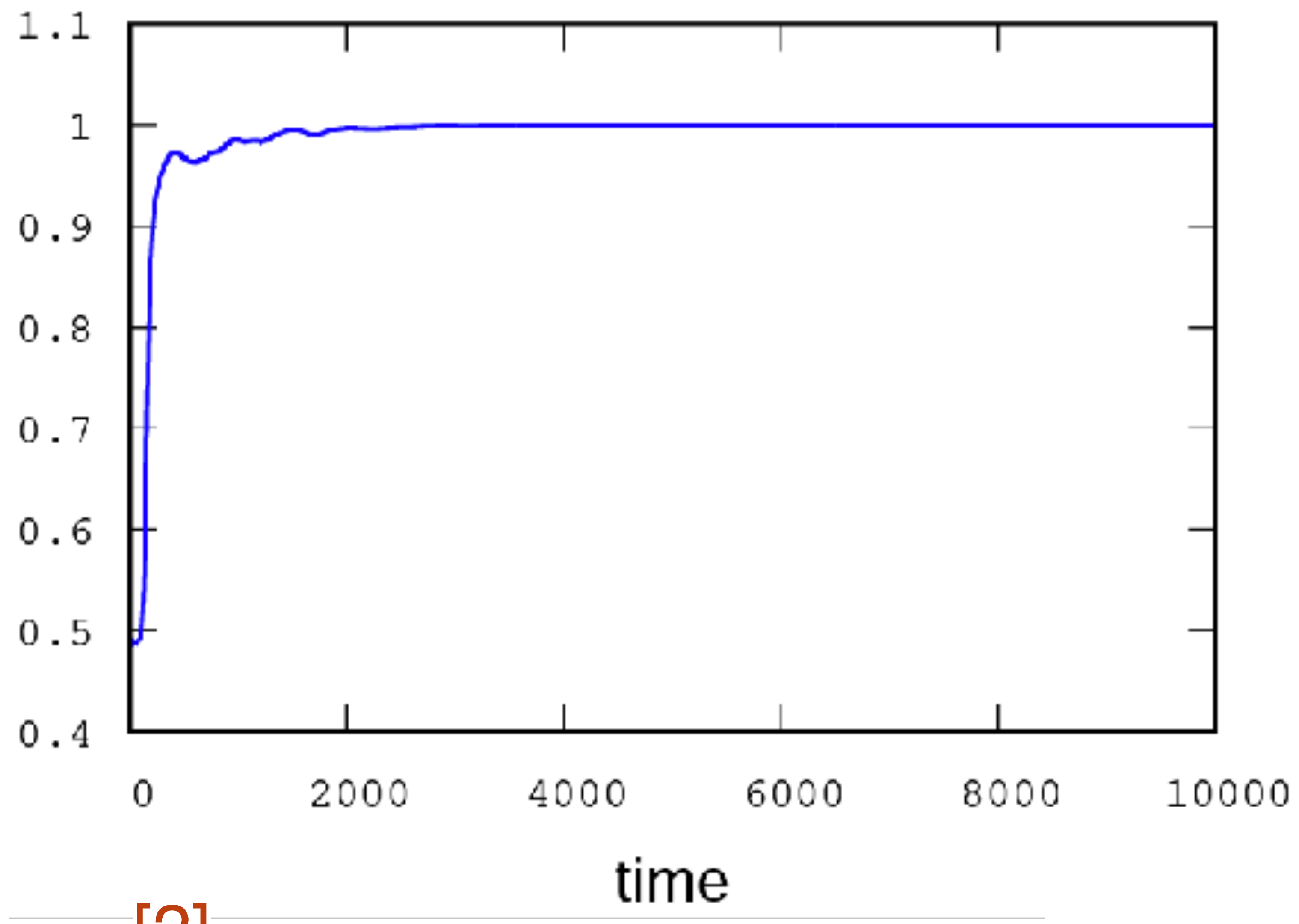
Unstable configuration C (Dispersion)

Minimum value of lapse function α



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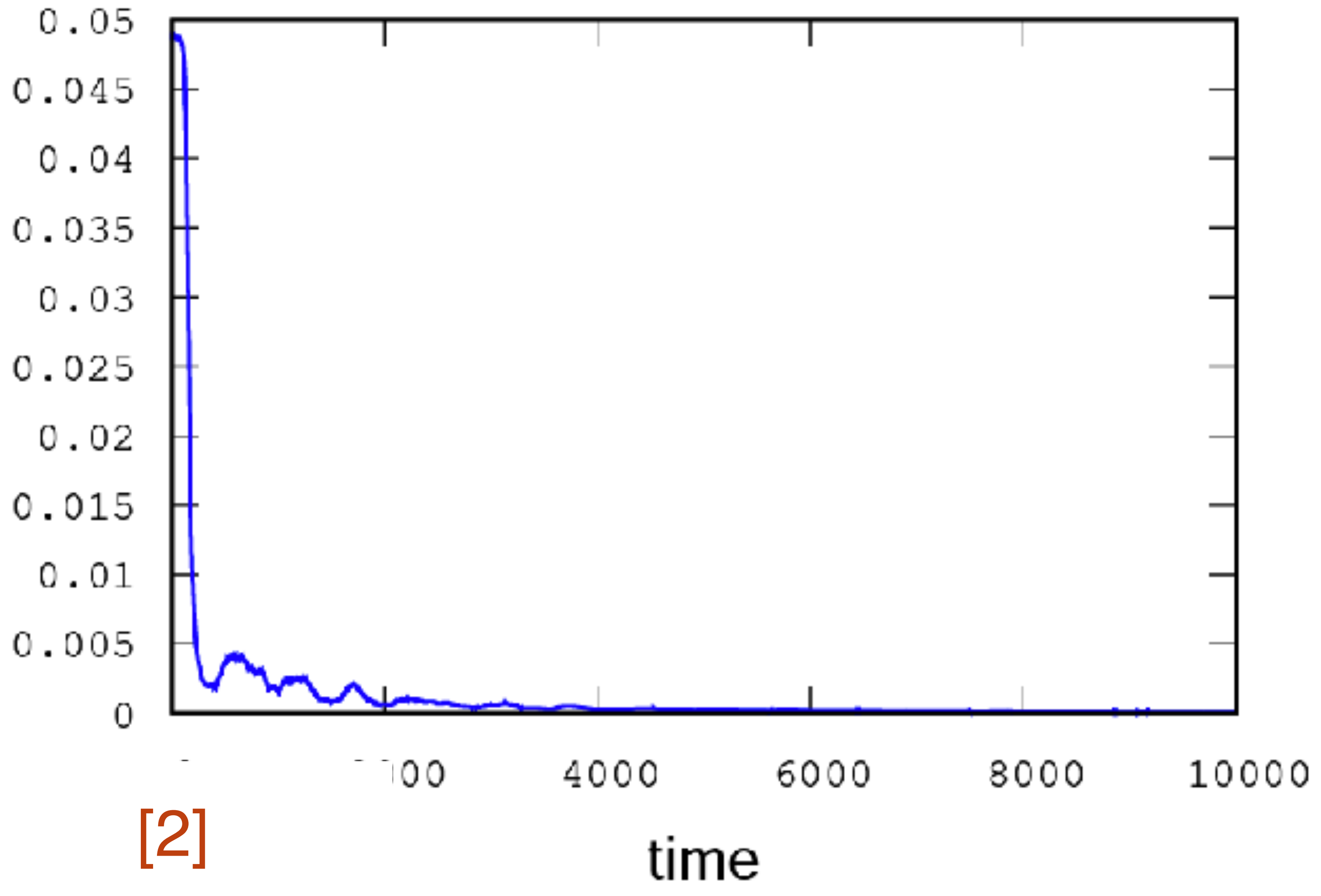
Maximum norm of scalar field $|\varphi|$



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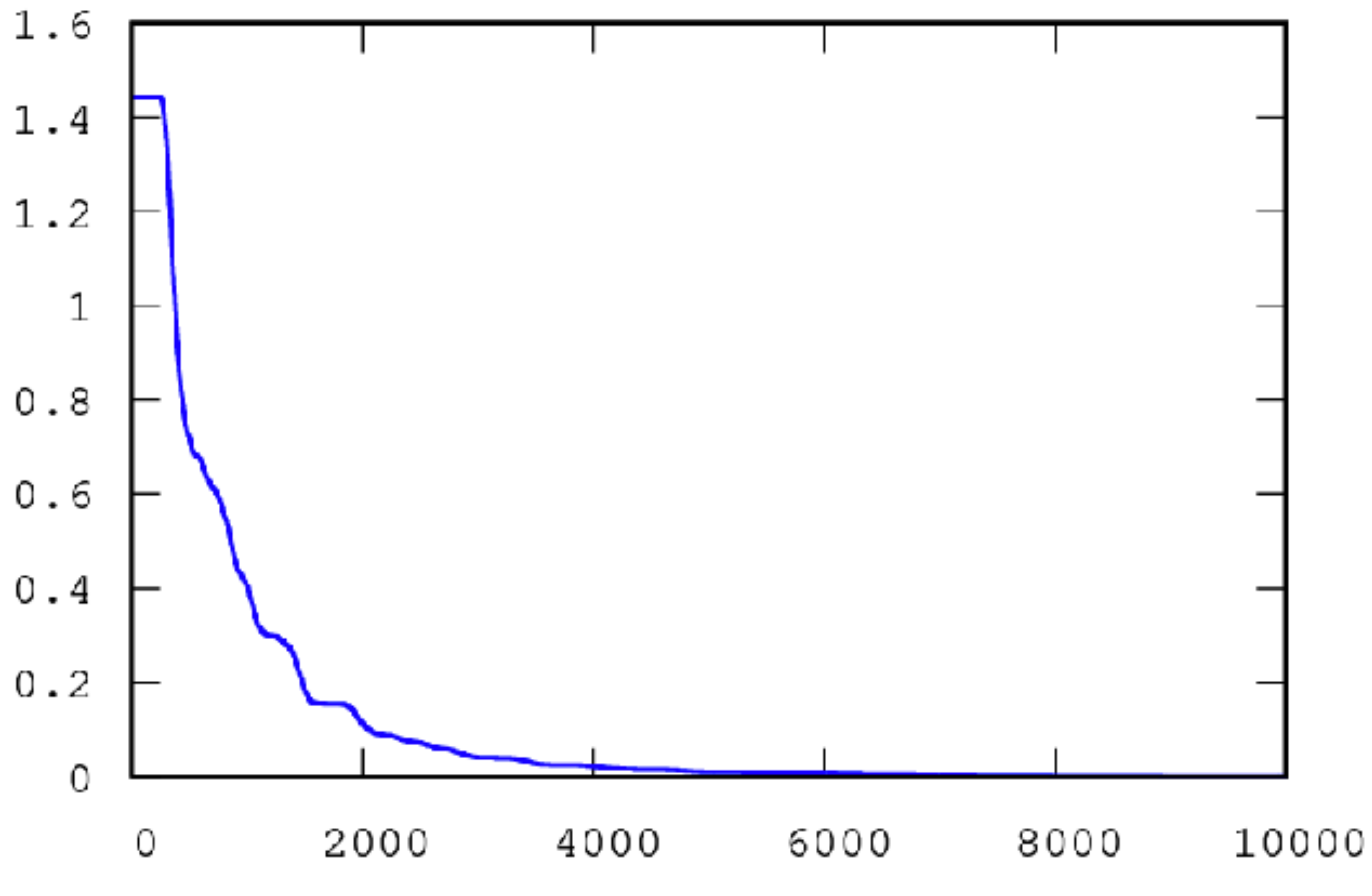
Mass M at boundary



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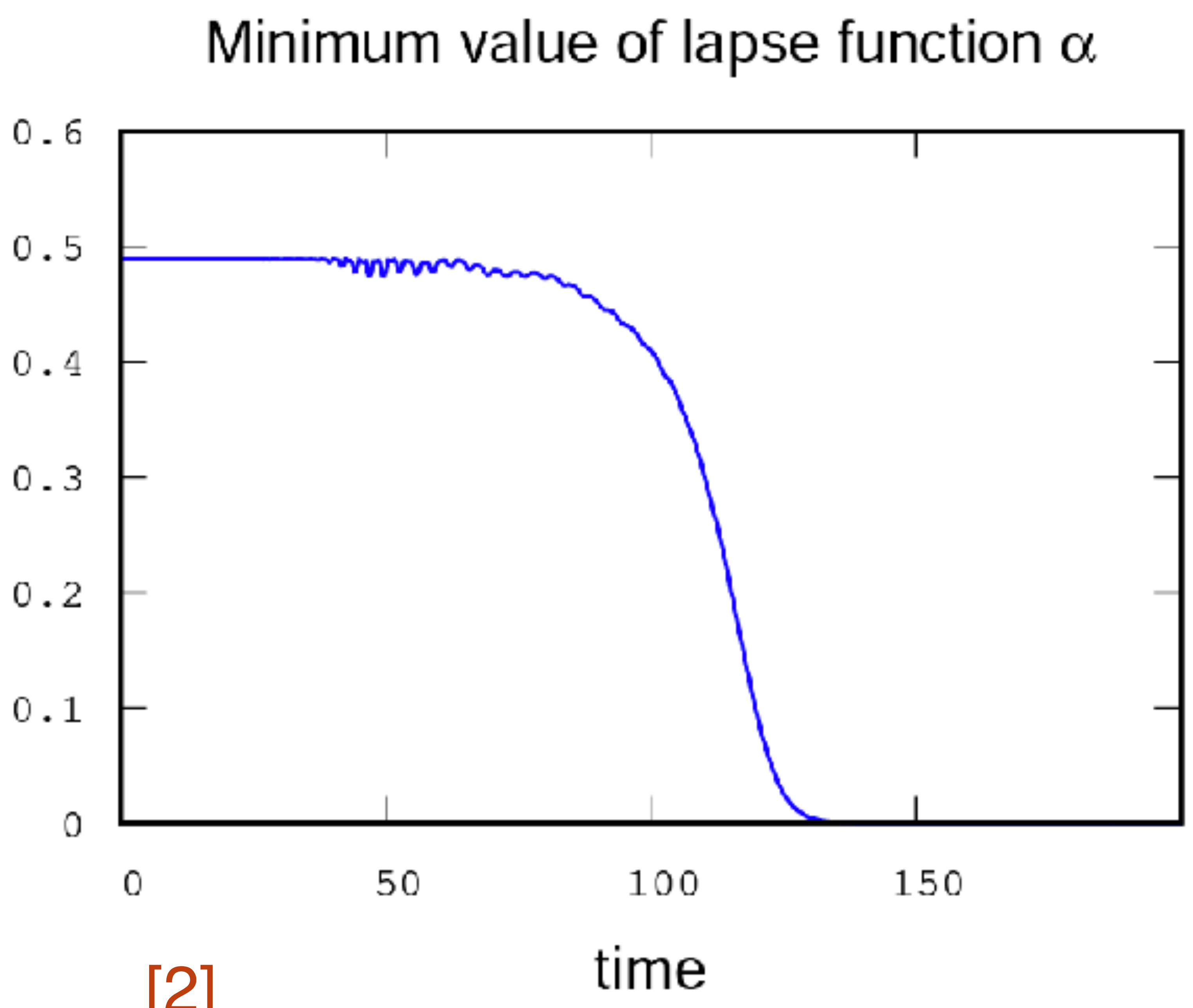
Unstable configuration D (BH-Formation)



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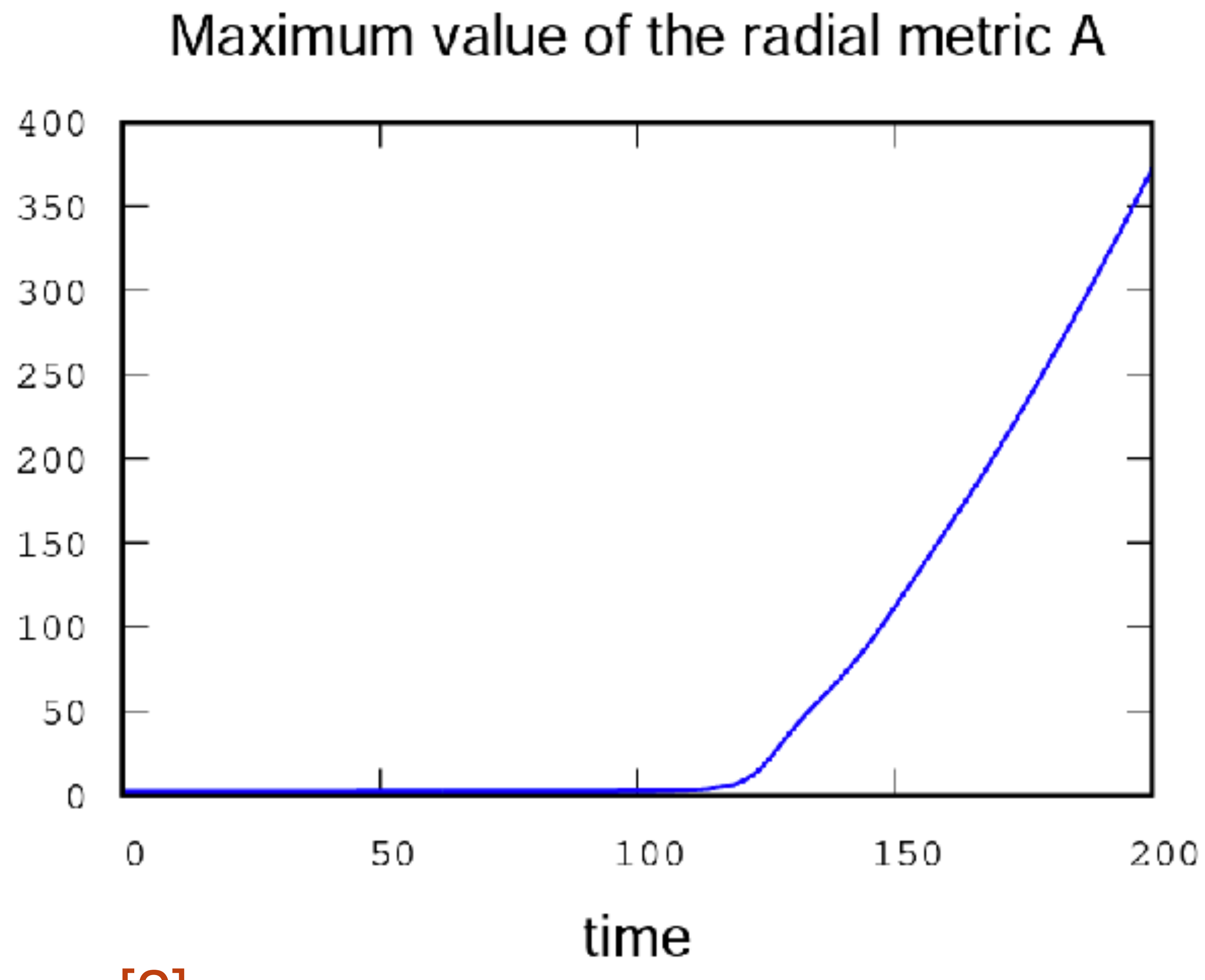

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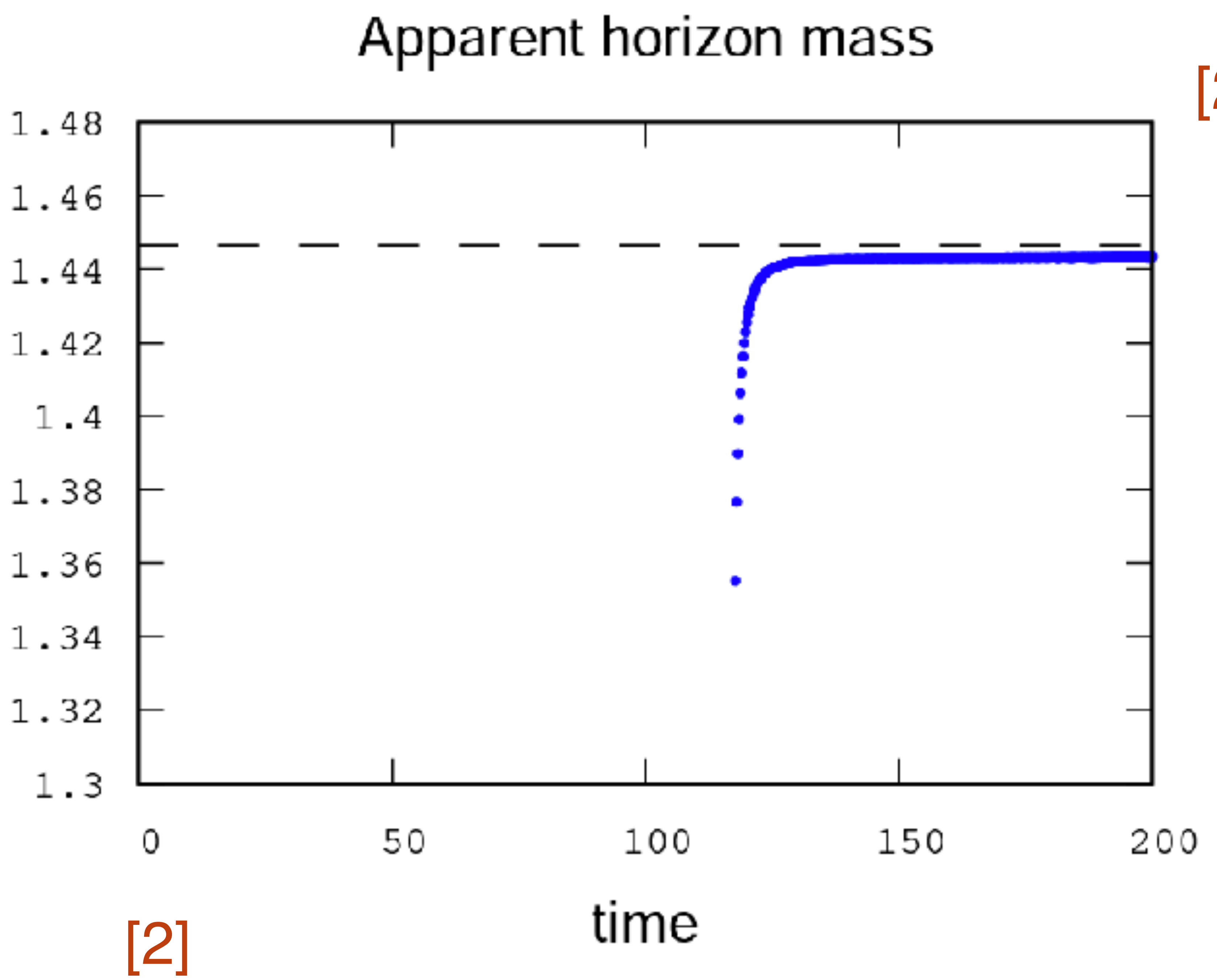

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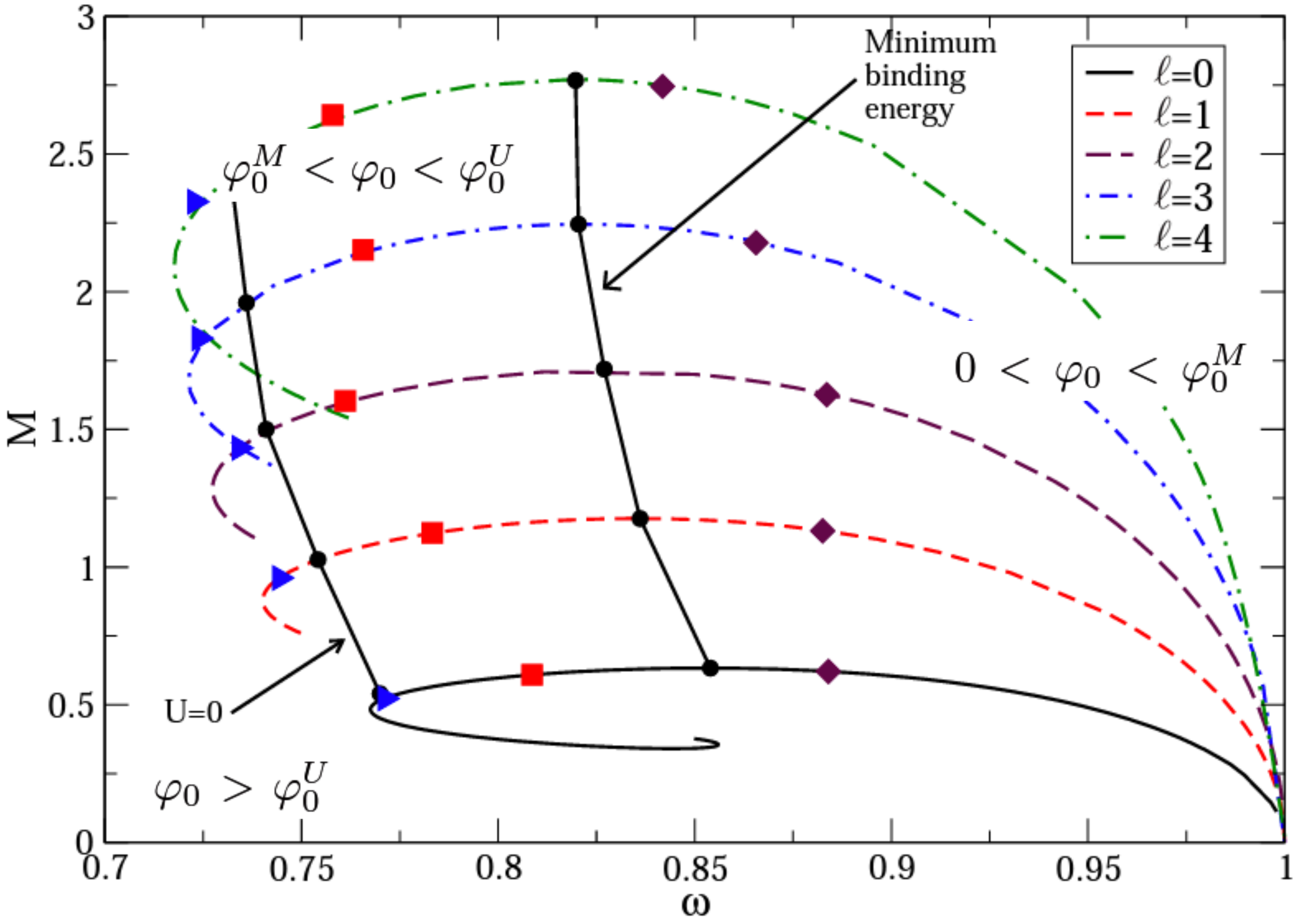
Final states

The region $0 < \varphi_0 < \varphi_0^M$ corresponds to bound stable configurations.
For all types of (small) perturbations studied, these configurations oscillate around the stationary solution.

The region $\varphi_0^M < \varphi_0 < \varphi_0^U$ corresponds to unstable but bound configurations that, depending on the specific type of perturbation, can either collapse to form a black hole or “migrate” to the stable branch.

This migration to the stable branch is achieved by ejecting excess scalar field to infinity.

The region $\varphi_0 > \varphi_0^U$ corresponds to unstable and unbound solutions that, depending on the specific type of perturbation, can either collapse to a black hole or dissipate to infinity



[2]

φ_0^M correspond to the maximum mass.

φ_0^U for which the binding energy is zero.

Stability with respect to non-radial perturbations

We test the stability of ℓ -BSs against non-spherical perturbations by performing numerical evolutions of the Einstein-Klein-Gordon system, in 3D.

We have considered non-spherical perturbations on the energy density of the form

$$\rho = \rho_0 \left[1 + \kappa \left(\frac{x^2 - y^2}{R_{99}^2}\right)\right]$$

We monitor the deformation parameters $\eta_y := \frac{I_{xx} - I_{zz}}{I_{xx} + I_{zz}}$, $\eta_z := \frac{I_{xx} - I_{yy}}{I_{xx} + I_{yy}}$,

where $I_{xx} = \int \rho(y^2 + z^2)\, dV$, $I_{yy} = \int \rho(x^2 + z^2)\, dV$

Those configurations known to be unstable under spherical perturbations, are also unstable under more general perturbations.

Stability with respect to non-radial perturbations

No growing modes have been measured in our simulations.

In this sense boson stars are stable against non-spherical perturbations.

For the timescales explored, the ℓ -BS belonging to the spherical stable branch do not exhibit measurable growing modes.

We find evidence of zero modes; that is, non-spherical perturbations that neither grow nor decay.

Configurations with large ℓ

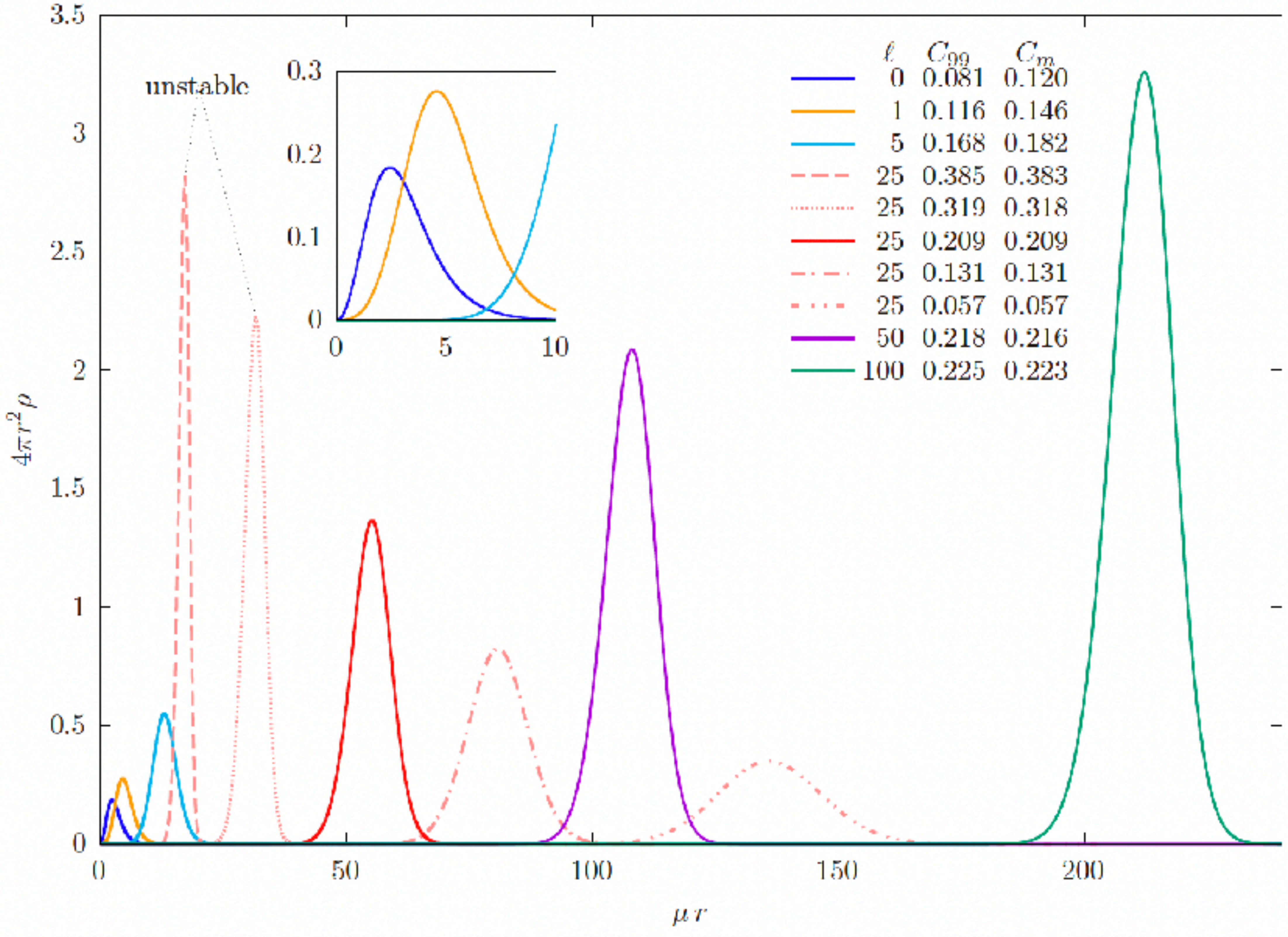
Besides the total mass and radii, the density and compactness are good characteristics to describe the configurations. We further also use the pressure.

$$\rho := -T^t{}_t = \frac{\kappa_\ell}{8\pi}\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 + \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right],$$

$$p_r := T^r{}_r = \frac{\kappa_\ell}{8\pi}\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 - \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right],$$

$$p_T := T^\theta{}_\theta = T^\varphi{}_\varphi = \frac{\kappa_\ell}{8\pi}\left[-\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 - \mu^2\psi_\ell^2\right].$$

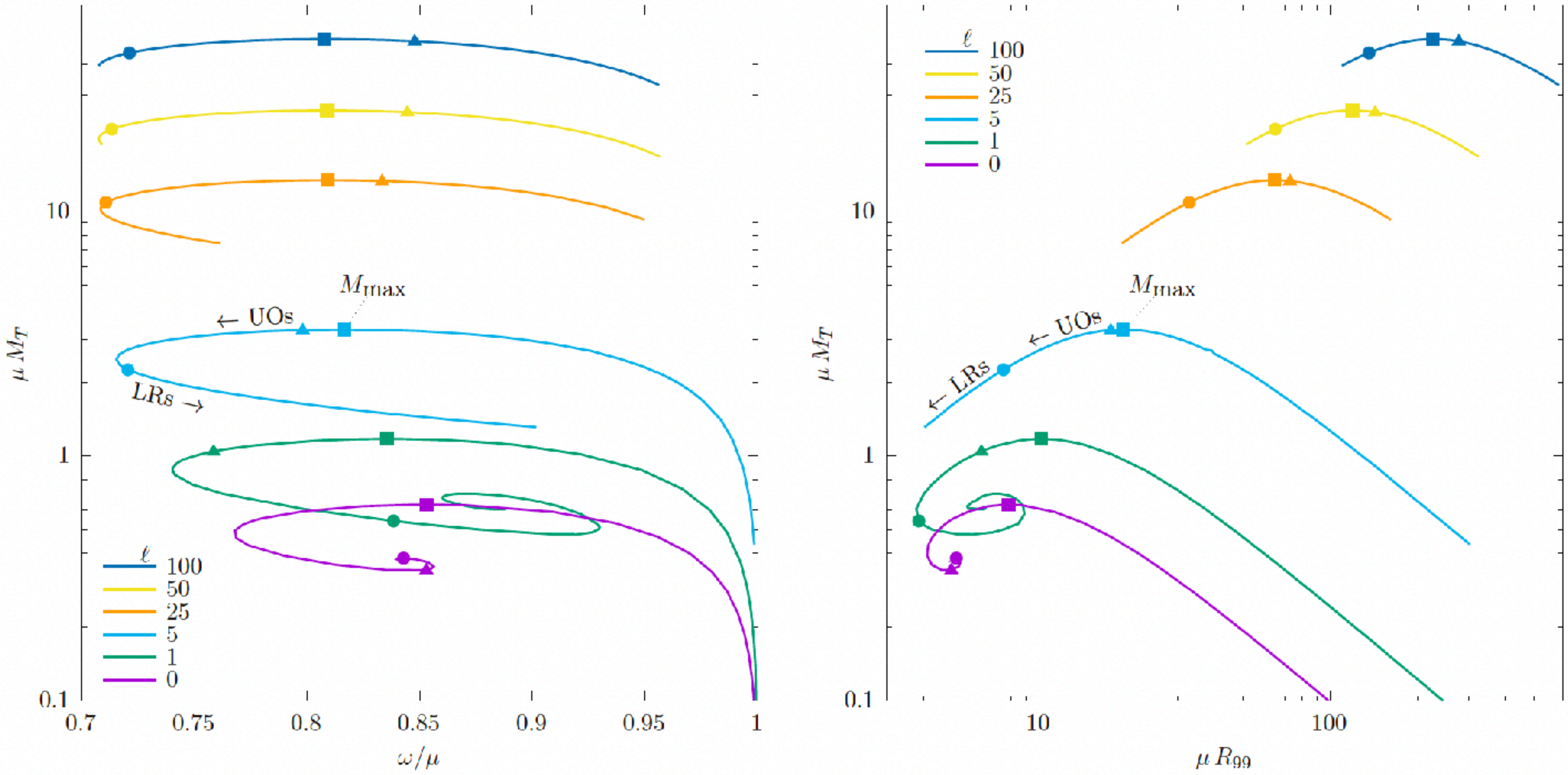
The compactness $$C_{99} := \frac{M_T}{R_{99}},$$



weighted density

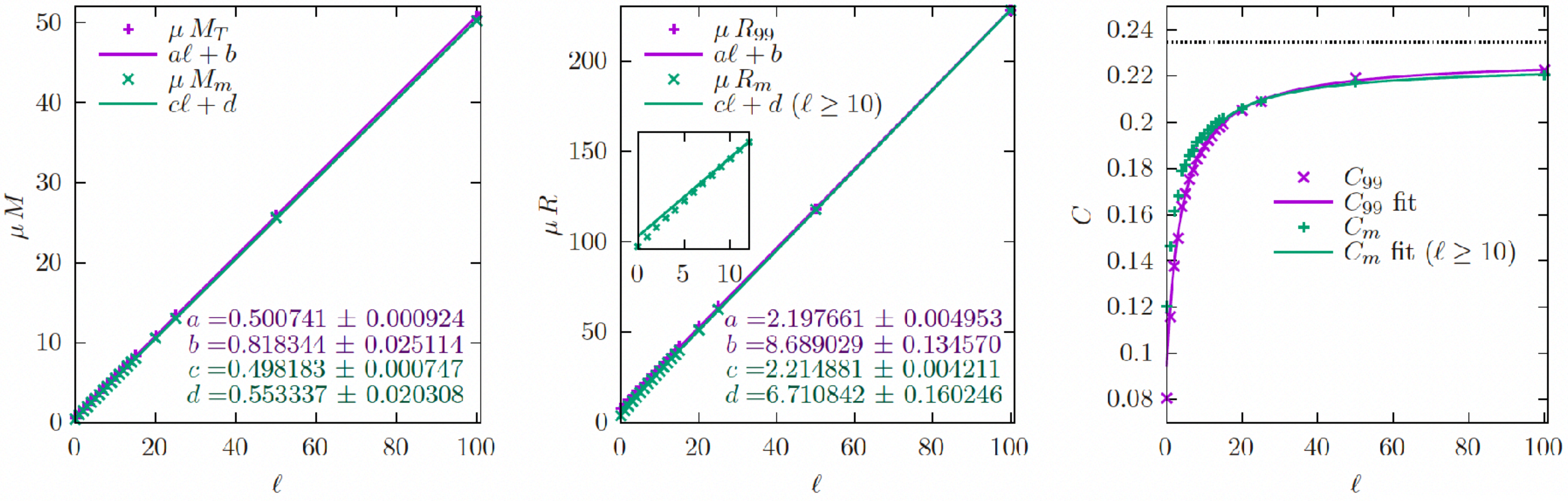
Configurations with large ℓ

The circles denote the first appearance of light rings (LR), while the triangles denote the first appearance of an ISCO-OSCO pair and, hence, the existence of unstable orbits (UOs).



Configurations with large ℓ

The total mass and radius grow linearly with ℓ. The compactness remain bounded, much below the Buchdal's limit



$$C_{99} := \frac{M_T}{R_{99}},$$

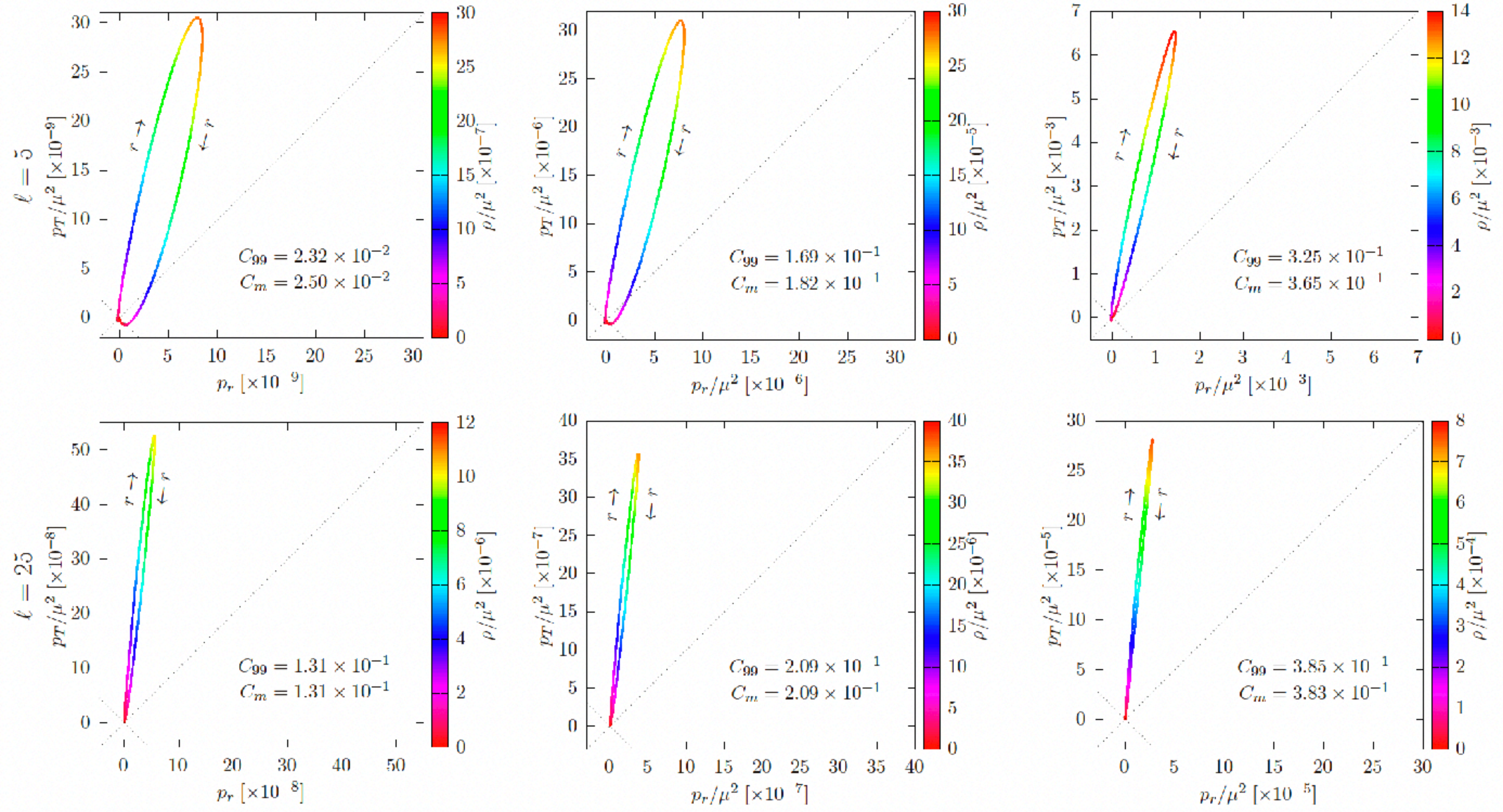
Configurations with large ℓ

The radial and tangential pressure is different in boson stars. The anisotropy grows with ℓ

$$p_r := T^r{}_r = \frac{\kappa_\ell}{8\pi}\left[\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 - \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right)\psi_\ell^2\right], \qquad p_T := T^\theta{}_\theta = T^\varphi{}_\varphi = \frac{\kappa_\ell}{8\pi}\left[-\frac{\psi_\ell'^2}{\gamma^2} + \frac{\omega^2}{\alpha^2}\psi_\ell^2 - \mu^2\psi_\ell^2\right].$$

Configurations with large ℓ

As ℓ increases, the "shells" become larger, as well as thinner relative to their radius, in such a way that the tangential pressure has to become larger relative to the radial one in order to support the configuration.



Configurations with large ℓ

Scaling of the solutions with $\omega = 0.8612$, for the quantities $M(r)/\ell$, γ and α.

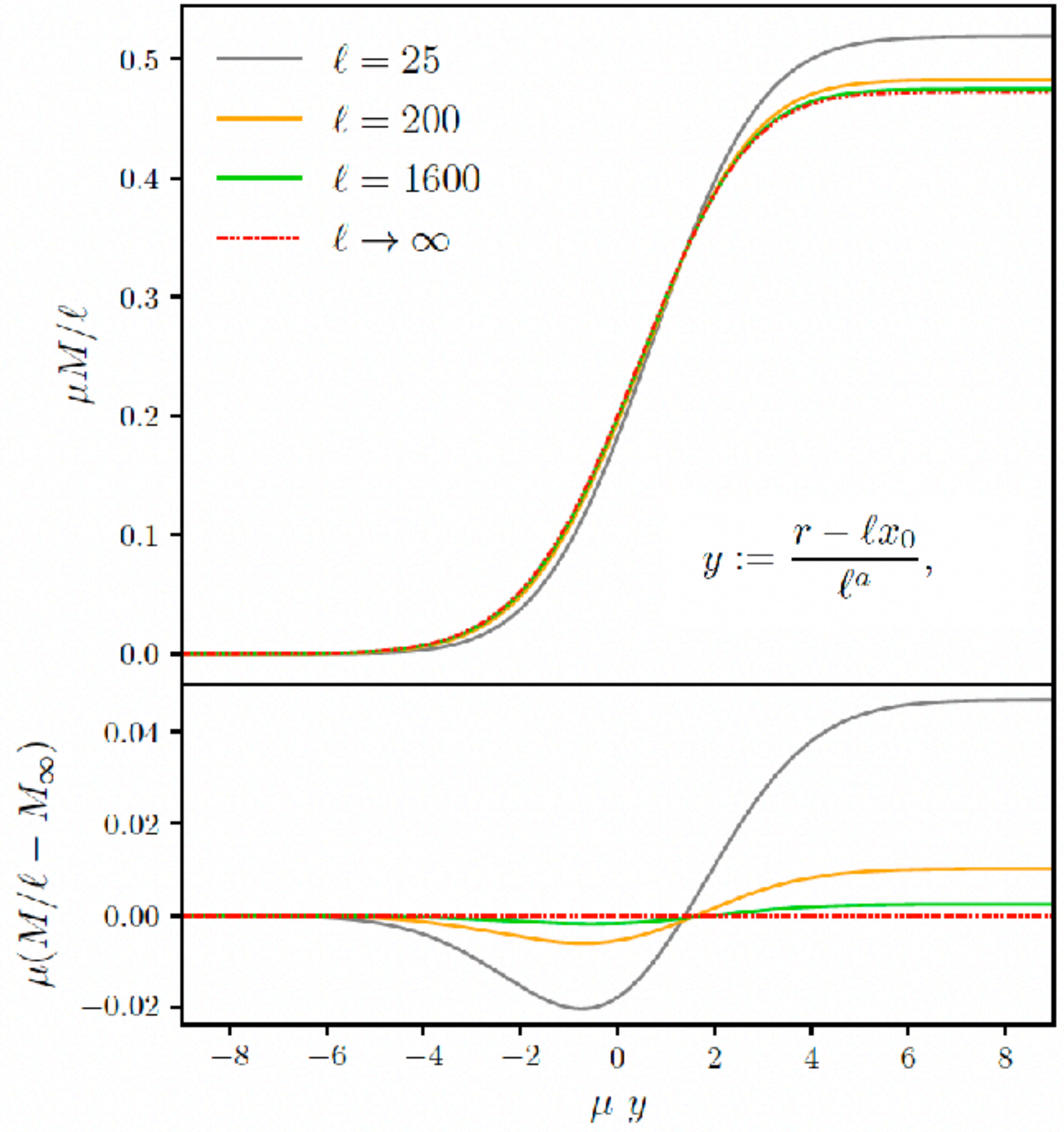
Configurations in the limit of large ℓ

By rescaling the fields (*M, α, γ, ψ*) and by shifting and rescaling the radial coordinate *r* in an appropriate way (which is largely motivated by the empirical numerical data and trial-and-error) one obtains a set of effective field equations which can be solved separately in the limit $\ell \to \infty$.

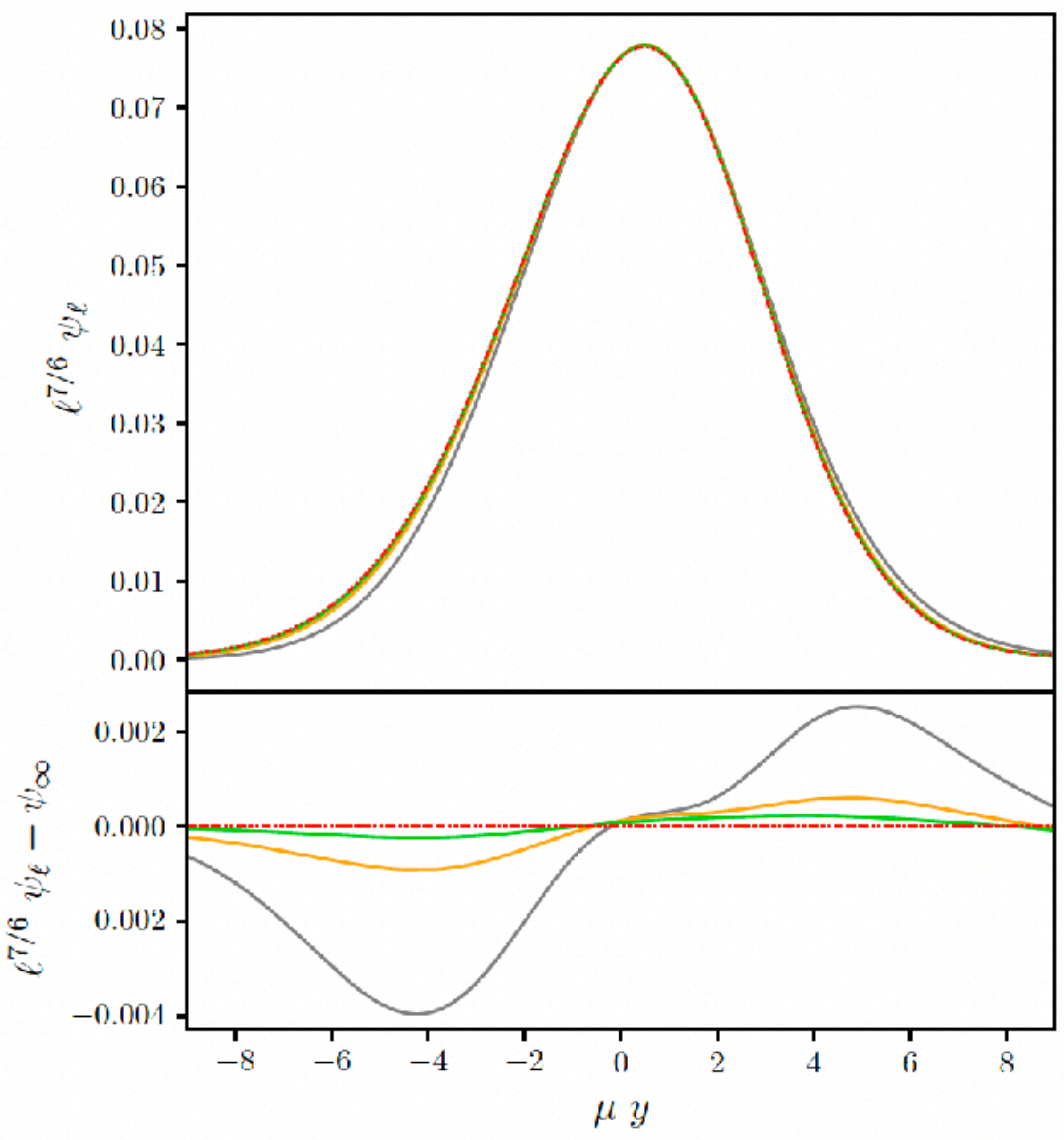
$$\frac{dM_\infty}{dy} = x_0^2 \rho_\infty, \qquad \rho_\infty = \frac{1}{\gamma_\infty^2}\left(\frac{d\psi_\infty}{dy}\right)^2 + \left(\frac{\omega^2}{\alpha_\infty^2} + \mu_0^2\right)\psi_\infty^2,$$

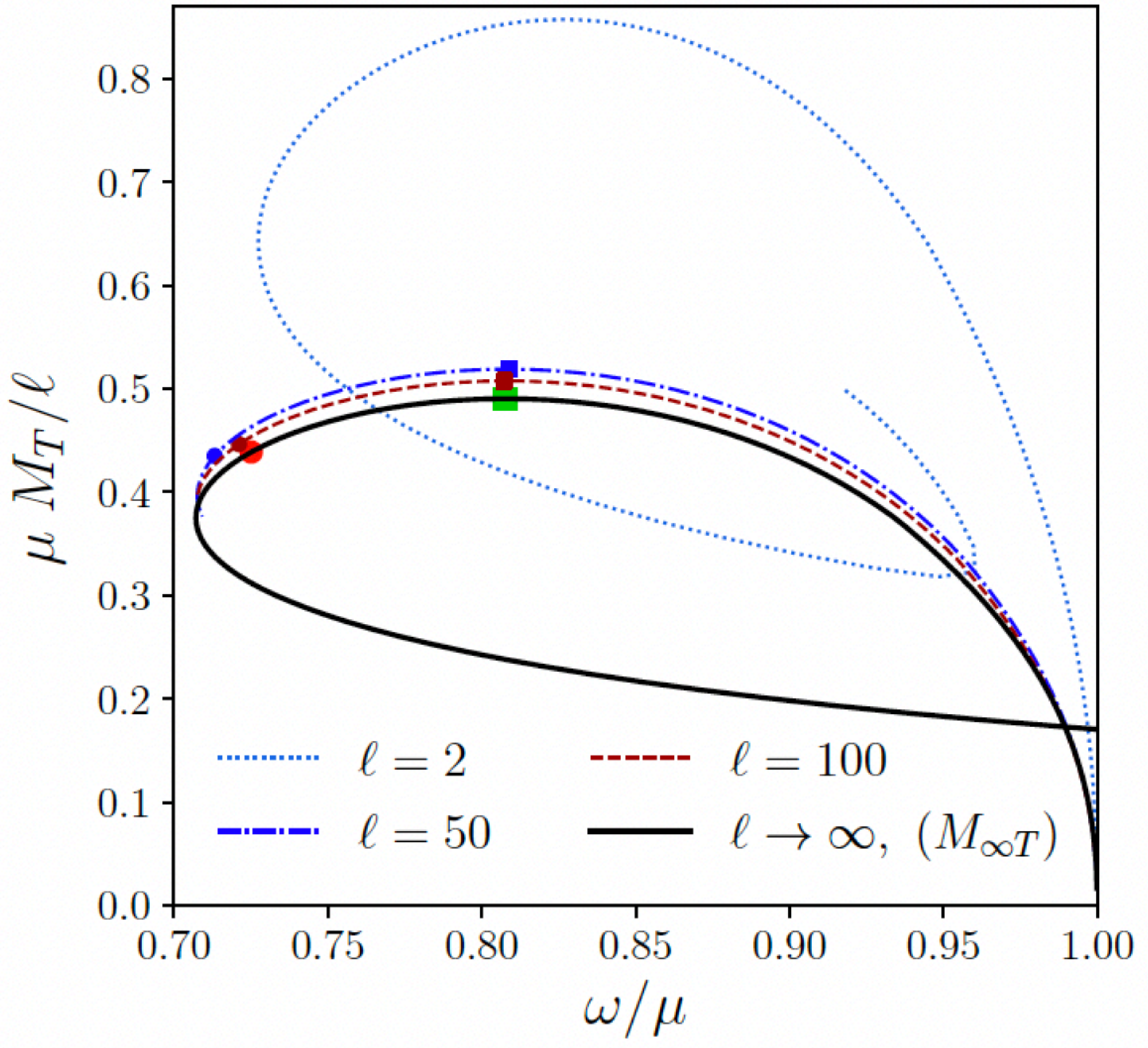
$$\frac{1}{\gamma_\infty^2 \alpha_\infty}\frac{d\alpha_\infty}{dy} = x_0 p_{r\infty}, \qquad p_{r\infty} = \frac{1}{\gamma_\infty^2}\left(\frac{d\psi_\infty}{dy}\right)^2 + \left(\frac{\omega^2}{\alpha_\infty^2} - \mu_0^2\right)\psi_\infty^2,$$

$$\frac{1}{\alpha_\infty \gamma_\infty}\frac{d}{dy}\left(\frac{\alpha_\infty}{\gamma_\infty}\frac{d\psi_\infty}{dy}\right) = -\left(\frac{\omega^2}{\alpha_\infty^2} - \mu_0^2\right)\psi_\infty,$$



Configurations in the limit of large ℓ





$$y := \frac{r - \ell x_0}{\ell^a},$$

Semiclassical approach

The semiclassical Einstein- Klein-Gordon system for a single real quantum scalar field whose state describes the excitation of N identical particles, each one corresponding to a given energy level, can be reduced to the Einstein- Klein-Gordon system for N complex classical scalar fields.

In the spherically symmetric and static scenario, where energy levels are labeled by quantum numbers n, ℓ and m and when all particles are accommodated in the ground state $n = \ell = m = 0$, one recovers the standard static boson star solutions, that can be excited if $n \neq 0$.

For the case where all particles have fixed radial and total angular momentum numbers n and ℓ, but are homogeneously distributed with respect to their magnetic number m, one obtains the ℓ-boson stars, whereas when $\ell = m = 0$ and n takes multiple values, multi-state boson star solutions are obtained.

Configurations with large ℓ

The starting point are Einstein's equations sourced by the expectation value of the stress energy-momentum tensor.

$$G_{\mu\nu} = 8\pi G\langle \hat{T}_{\mu\nu}\rangle.$$

Where in our case

$$\hat{T}_{\mu\nu} = (\nabla_\mu\hat{\phi})(\nabla_\nu\hat{\phi}) - \frac{1}{2}g_{\mu\nu}\left[(\nabla_\alpha\hat{\phi})(\nabla^\alpha\hat{\phi}) + {m_0}^2\hat{\phi}\hat{\phi}\right].$$

For a static an spherically symmetric spacetime

$$ds^2 = -\alpha^2(\vec{x})dt^2 + \gamma_{ij}(\vec{x})dx^i dx^j,$$

Configurations with large ℓ

Besides the total mass and radii, the density and compactness are good characteristics to describe the configurations. We further also use the pressure.

$$R^{(3)} = 16\pi G\rho,$$

$$R^{(3)}_{ij} - \frac{1}{\alpha} D_i D_j \alpha = 4\pi G \left[\gamma_{ij}(\rho - S) + 2S_{ij}\right],$$

$$\rho := n^\mu n^\nu \langle \hat{T}_{\mu\nu} \rangle,$$

$$S_{ij} := (\delta_i{}^\mu + n_i n^\mu)(\delta_j{}^\nu + n_j n^\nu)\langle \hat{T}_{\mu\nu} \rangle$$

Choosing the states

$$\hat{\phi}(x) = \sum_{n\ell m} \frac{1}{\sqrt{2\omega_{n\ell}}} \left[\hat{a}_I e^{-i\omega_{n\ell}t} v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi) + \mathrm{H.c.}\right].$$

The scalar field quantities written in terms of the operators.

$$\begin{aligned}\rho \ = \sum_{I,J} \frac{1}{4\sqrt{\omega_I \omega_J}} \Big\{ \langle \hat{a}_I \hat{a}_J \rangle \Big[\Big(-\frac{\omega_I \omega_J}{\alpha^2} + {m_0}^2 \Big) u_I u_J + (D_k u_I)(D^k u_J) \Big] e^{-i(\omega_I + \omega_J)t} \\ + \langle \hat{a}_I^\dagger \hat{a}_J \rangle \Big[\Big(+\frac{\omega_I \omega_J}{\alpha^2} + {m_0}^2 \Big) u_I^* u_J + (D_k u_I^*)(D^k u_J) \Big] e^{+i(\omega_I - \omega_J)t} + \mathrm{c.c.} \Big\},\end{aligned}$$

$$u_I(\vec{x}) = v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi), \quad I = (n\ell m),$$

For a state with a definite number of particles, it follows that.

$$\langle \hat{a}_I \hat{a}_J \rangle = 0 \text{ and } \langle \hat{a}_I^\dagger \hat{a}_J \rangle = N_I \delta_{IJ},$$

Boson stars family

We end up with a system of equations that reduces to the system on N classical complex scalar fields .

$$\psi''_{n\ell} = -\left[\gamma^2 + 1 - (2\ell+1)r^2\gamma^2\left(\frac{\ell(\ell+1)}{r^2} + {m_0}^2\right)(\psi_{n\ell})^2\right]\frac{\psi'_{n\ell}}{r} - \left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - {m_0}^2\right)\gamma^2\psi_{n\ell},$$

$$\gamma' = \sum_{n\ell}\frac{2\ell+1}{2}r\gamma\left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2} + {m_0}^2\right)\gamma^2(\psi_{n\ell})^2 + (\psi'_{n\ell})^2\right] - \left(\frac{\gamma^2-1}{2r}\right)\gamma,$$

$$\alpha' = \sum_{n\ell}\frac{2\ell+1}{2}r\alpha\left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - {m_0}^2\right)\gamma^2(\psi_{n\ell})^2 + (\psi'_{n\ell})^2\right] + \left(\frac{\gamma^2-1}{2r}\right)\alpha,$$

For the function

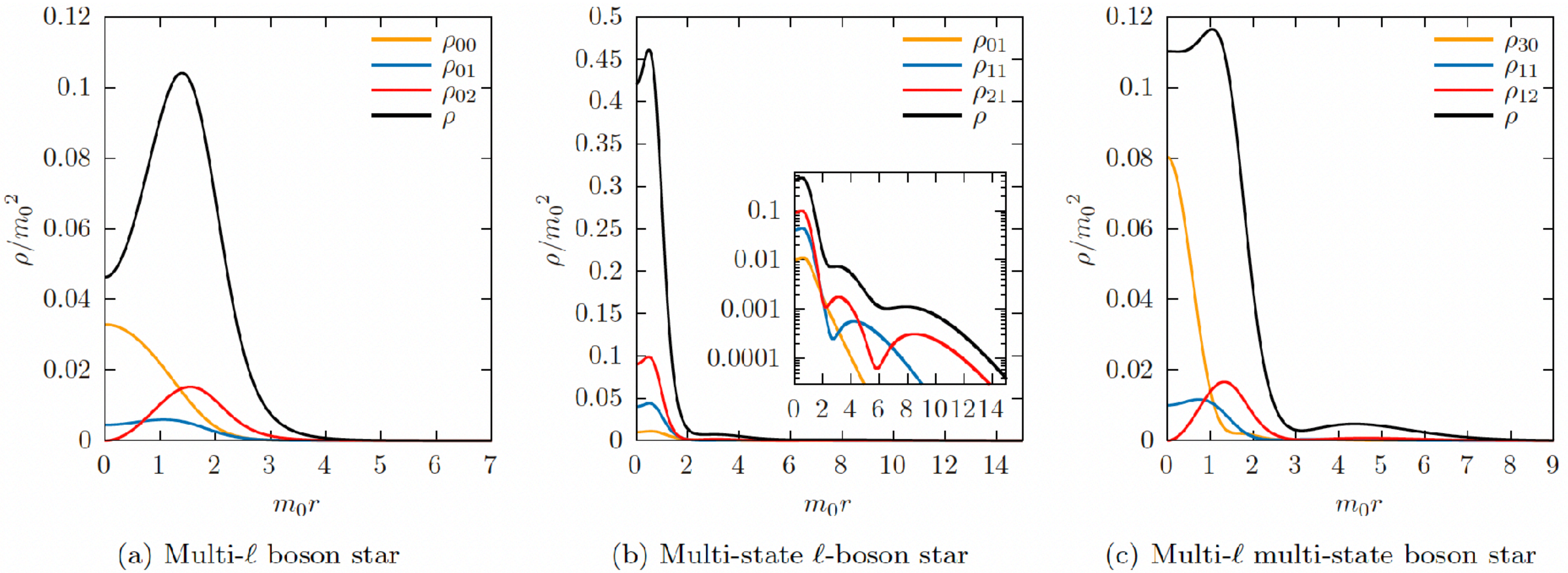
$$\psi_{n\ell} = \sqrt{\frac{N_{n\ell m}}{\omega_{n\ell}}}v_{n\ell}.$$

Boson stars family

ℓ - Boson stars correspond to a particular excitation of a single real quantum spin zero field that describes a selfgravitating system of $(2\ell+1)N_{0\ell m}$ identical quantum particles.

Name	n	ℓ
Multi-ℓ multi-state boson star	$n_1, n_2, \ldots, n_p$	$\ell_1, \ell_2, \ldots, \ell_q$
Multi-state ℓ-boson star	$n_1, n_2, \ldots, n_p$	ℓ_1
Multi-state boson star	$n_1, n_2, \ldots, n_p$	0
Boson star	n_1	0
ℓ-Boson star	n_1	ℓ_1
Multi-ℓ boson star	n_1	$\ell_1, \ell_2, \ldots, \ell_q$

Boson stars family



Conclusions

* In ℓ-boson stars as the value of ℓ grows, one finds more massive and compact stable objects.

* Stable configurations, when perturbed, oscillate around the unperturbed solution and seem to very slowly return to a stationary configuration.

* Unstable configurations, can have three different final states: collapse to a black hole, migration to the stable branch, or explosion (dissipation) to infinity.

* As ℓ grows, so does the object and also the almost empty central region, tending to form shells of scalar fields where the size of the almost empty central region is much larger than the size of the region where the scalar field is mainly distributed.

* The mass of the solutions that divide the stable and the unstable branches, as well as their size, grows with ℓ, but in such a way that the compactness tends to a finite value. In the $\ell \to \infty$ limit the compactness tends to about 0.23 for the maximum mass configuration; that is, about half the Buchdahl limit.

* By using the semiclassical approach we showed that ℓ-boson stars are just a set of a larger family of solutions of the Einstein-Klein Gordon system in spherical symmetry.

Thank you, It is always a pleasure for me to come to Belém