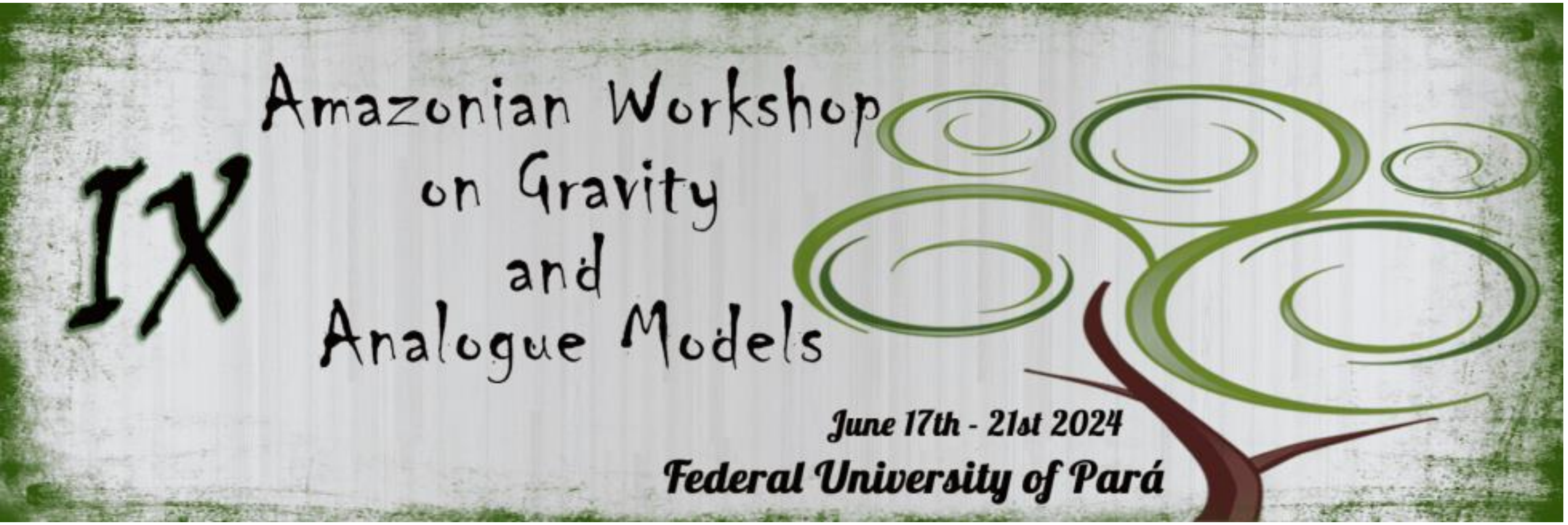
Compact Objects and How to Model Them
Part I



Guilherme Raposo

Universidade de Aveiro (Gr@v)

17/06/2024

Compact Object Course - Overview

Our Goals:

- Understand the frameworks to model **compact objects** and **ultracompact objects** in general relativity.

How will we do it?

- What are compact objects?
- Types of compact objects? How to model them?
- What are exotic compact objects? Why do we care?
- How to model them?
- How to test for them

Compact Object Course - Overview

Some time ago...

Compact Object Course - Overview

Some time ago...

Now I will try give you a (short) updated version!

TÉCNICO LISBOA

Tópicos em Relatividade Geral e Cosmologia

Course description

8 fevereiro 2016, 11:42 • Vincenzo Vitagliano

Introduction to QFT in curved spacetime (8 lectures) - Dr. Vincenzo Vitagliano

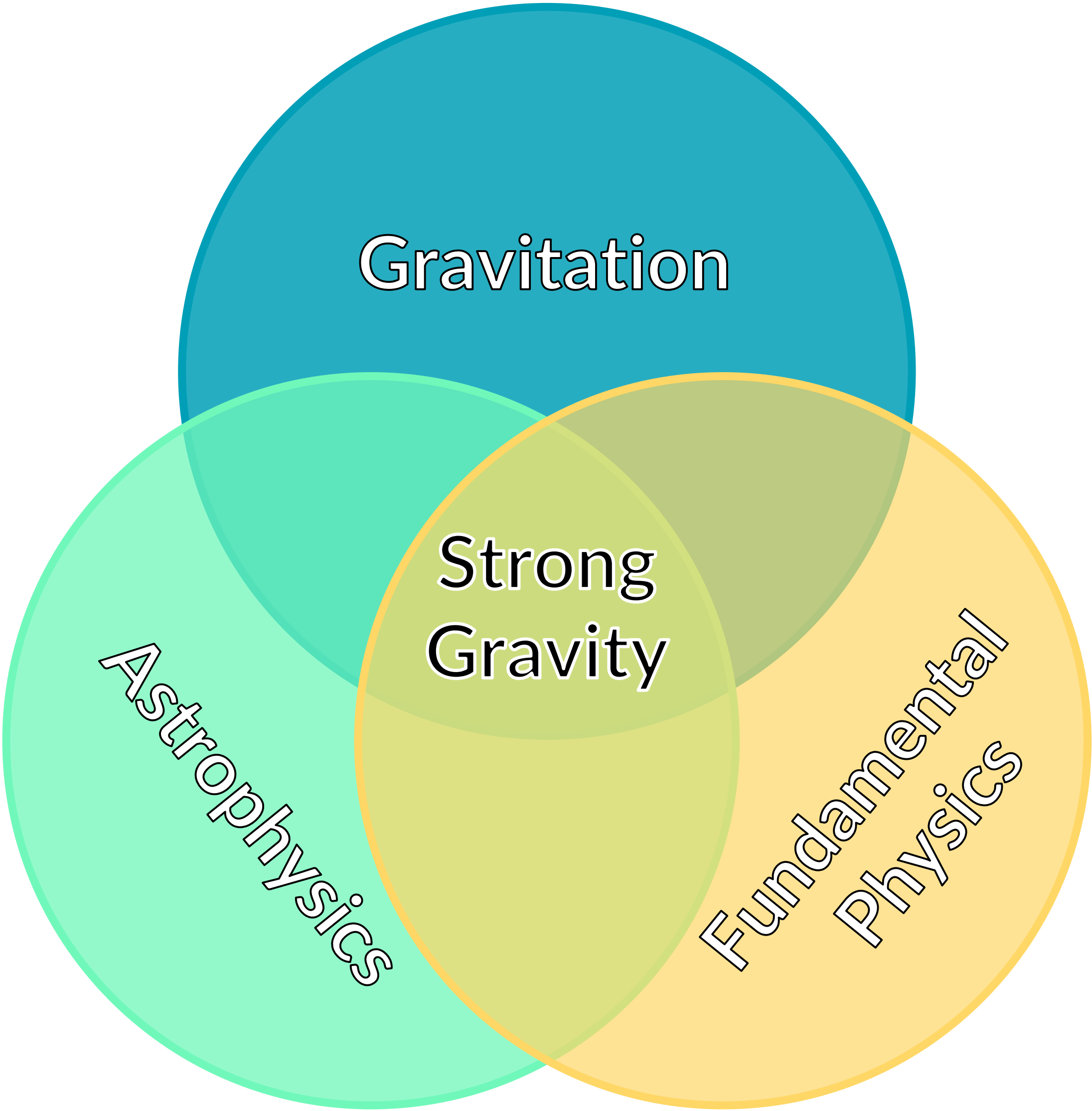
Canonical quantization and particle production. Quantum fields in an expanding universe. Bogoliubov transformations. Unruh effect. Hawking radiation and black holes thermodynamics. The Casimir effect. Path integrals and vacuum polarization. Effective action for a driven harmonic oscillator and in general. Semiclassical gravity. Zeta function renormalization. Computation of functions using heat kernels.

Compact objects (8 lectures) - Dr. Caio Macedo

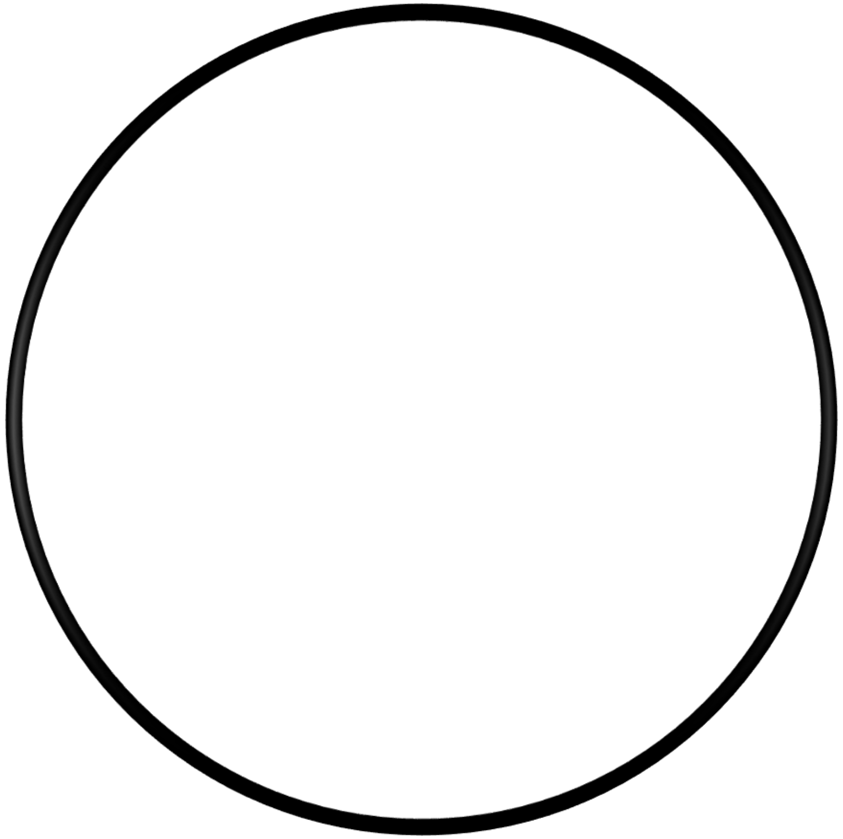
Newtonian stars and dark stars. Relativistic matter: equation of state. Equilibrium configurations. Basic properties of white dwarfs and neutron stars. Black holes: brief view. Stars formed by fundamental fields: boson stars, oscillatons, singlets.

Numerical methods for solving partial differential equations. Elliptic equations. Hyperbolic equations. Fixed and adaptive mesh refinement. The two-body problem in general relativity. 3+1 splitting of spacetime. Extrinsic curvature. The constraints. ADM evolution equations. BSSN evolution equations. Numerical solution of the constraints. Wave extraction and the Newman-Penrose formalism.

Compact Objects



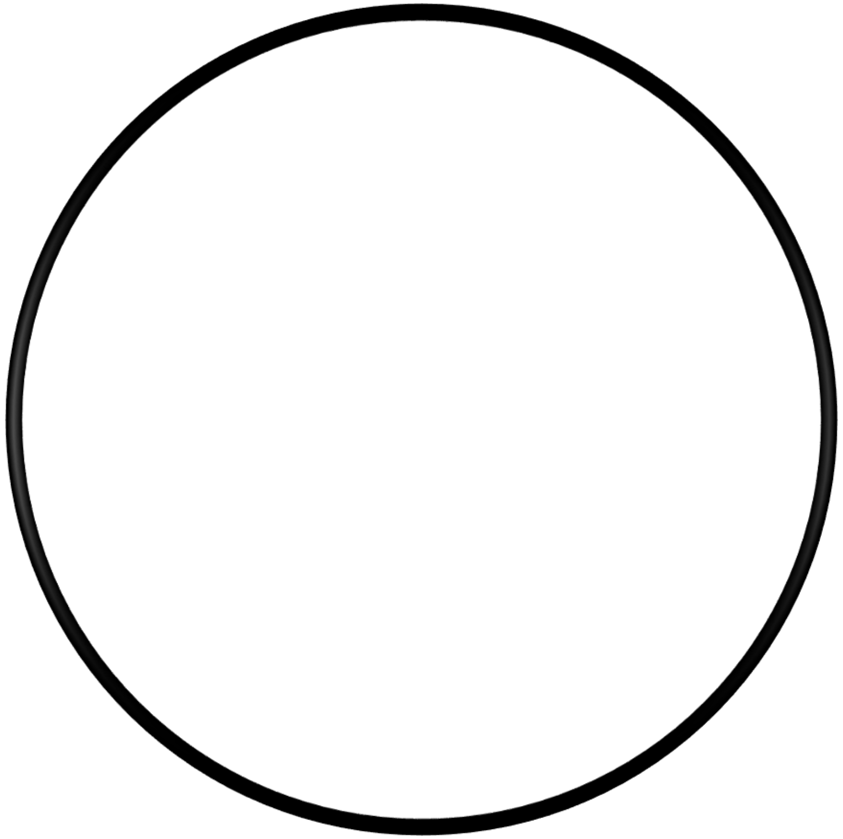
What defines how compact a celestial body is?



$$\vec{g} = -\frac{GM}{r^2}\vec{e}_r$$

Back to basics

What defines how compact a celestial body is?

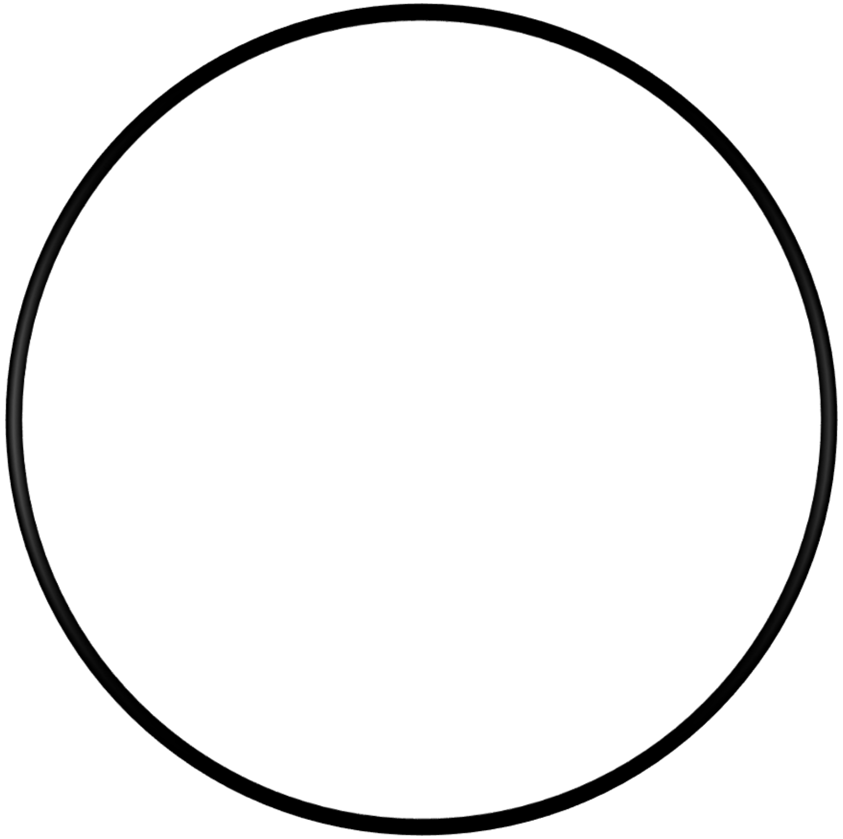


$$\vec{g} = -\frac{GM}{r^2}\vec{e_r}$$

What macroscopic quantities are needed to define the gravitational field of a finite-size body?

1. Mass

What defines how compact a celestial body is?

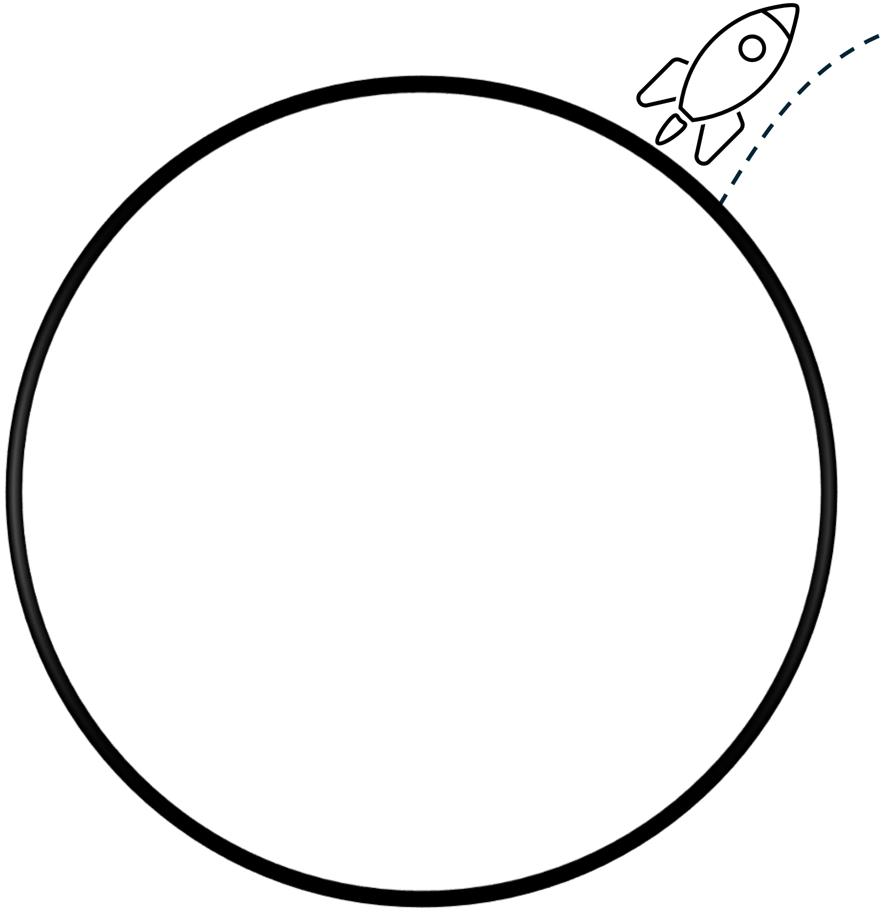


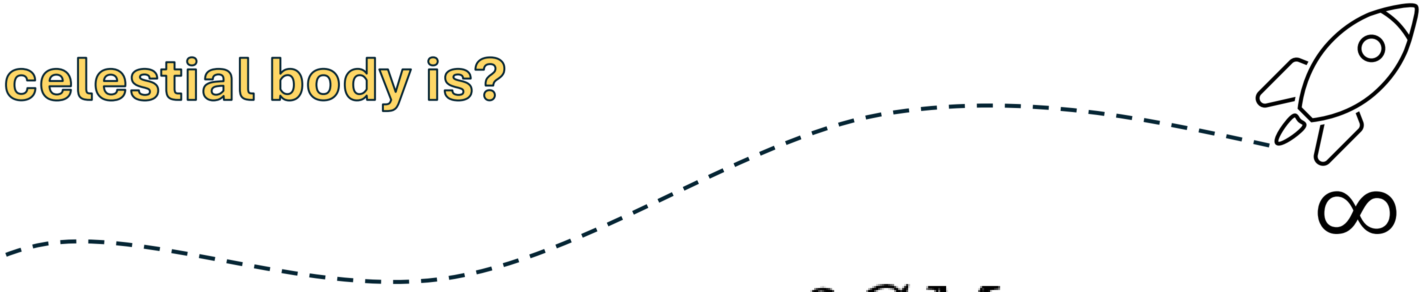
$$\vec{g} = -\frac{GM}{r^2}\vec{e_r}$$

What macroscopic quantities are needed to define the gravitational field of a finite-size body?

1. Mass
2. Radius

What defines how compact a celestial body is?





Escape velocity:

$$v^2 = \frac{2GM}{R}$$

What if light cannot escape?

"*Dark Star*" – John Mitchel, 1783

Dimensionless quantity:

$$C = \frac{GM}{c^2 R}$$

Compact Objects in Our Universe

Table 1.1

Distinguishing Traits of Compact Objects

Object	Mass[a] (M)	Radius[b] (R)	Mean Density (g cm^{-3})	Surface Potential (GM/Rc^2)
Sun	$M_\odot$	$R_\odot$	1	10^{-6}
White dwarf	$\lesssim M_\odot$	$\sim 10^{-2} R_\odot$	$\lesssim 10^7$	$\sim 10^{-4}$
Neutron star	$\sim 1\text{–}3 M_\odot$	$\sim 10^{-5} R_\odot$	$\lesssim 10^{15}$	$\sim 10^{-1}$
Black hole	Arbitrary	$2GM/c^2$	$\sim M/R^3$	~ 1

[a] $M_\odot = 1.989 \times 10^{33}$ g
[b] $R_\odot = 6.9599 \times 10^{10}$ cm

[Table 1.1: Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects, Shapiro & Teukolsky]

Compact Objects: A definition

Compact Object (CO):
Object who's exterior spacetime contains an ISCO.

$$R < 6M$$

$$r_0 < 3M$$

$$r_0 \lesssim 2.04M$$

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Object who's exterior spacetime contains a photonsphere.

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Compact object that is not a black hole nor a neutron star.

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Black Hole Mimicker:
Ultracompact object that is mimics the properties of a black hole.

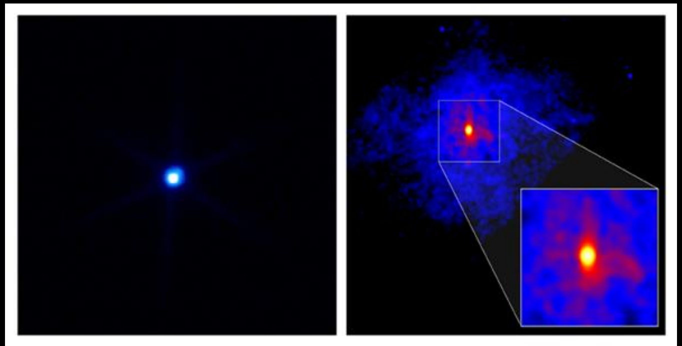
Compact Objects in Our Universe

< 8 Solar Masses

White Dwarf

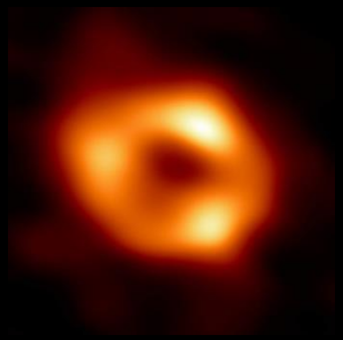
[8, 25] Solar Mass

Neutron Star



> 25 Solar Mass

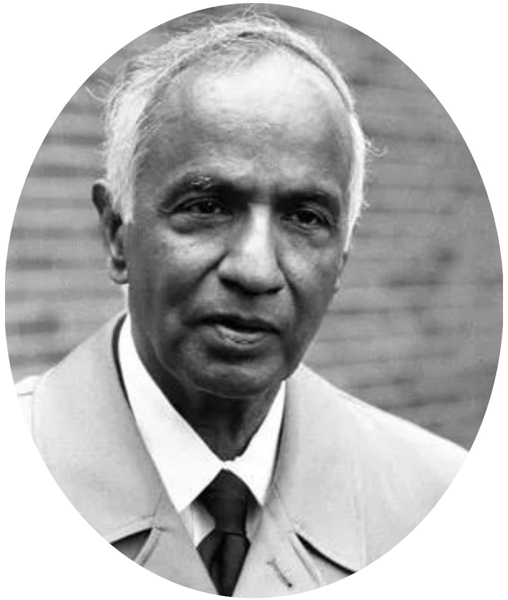
Black Hole



Black Holes

"In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, provides the absolute exact representation of untold numbers of massive black holes that populate the universe. "

S. Chandrasekhar,
The Nora and Edward Ryerson Lecture, Chicago 1975



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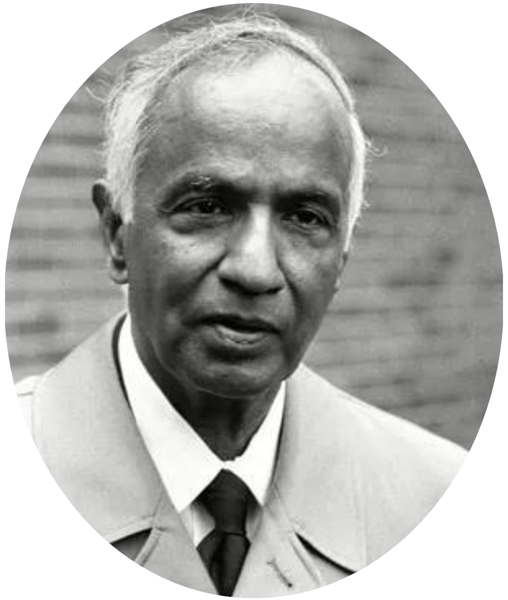


*Black Holes are **simple** and **economical**!*

***One** single **exact** solution:*
- *Stellar BHs*
- *Supermassive BHs*

Only requires two/three parameters
- *Mass, Angular Momentum, Charge*

Neutron Stars

Layered Structure:

Outer Crust:

- Coulomb lattice with heavy nuclei & degenerate electron gas

Inner Crust:

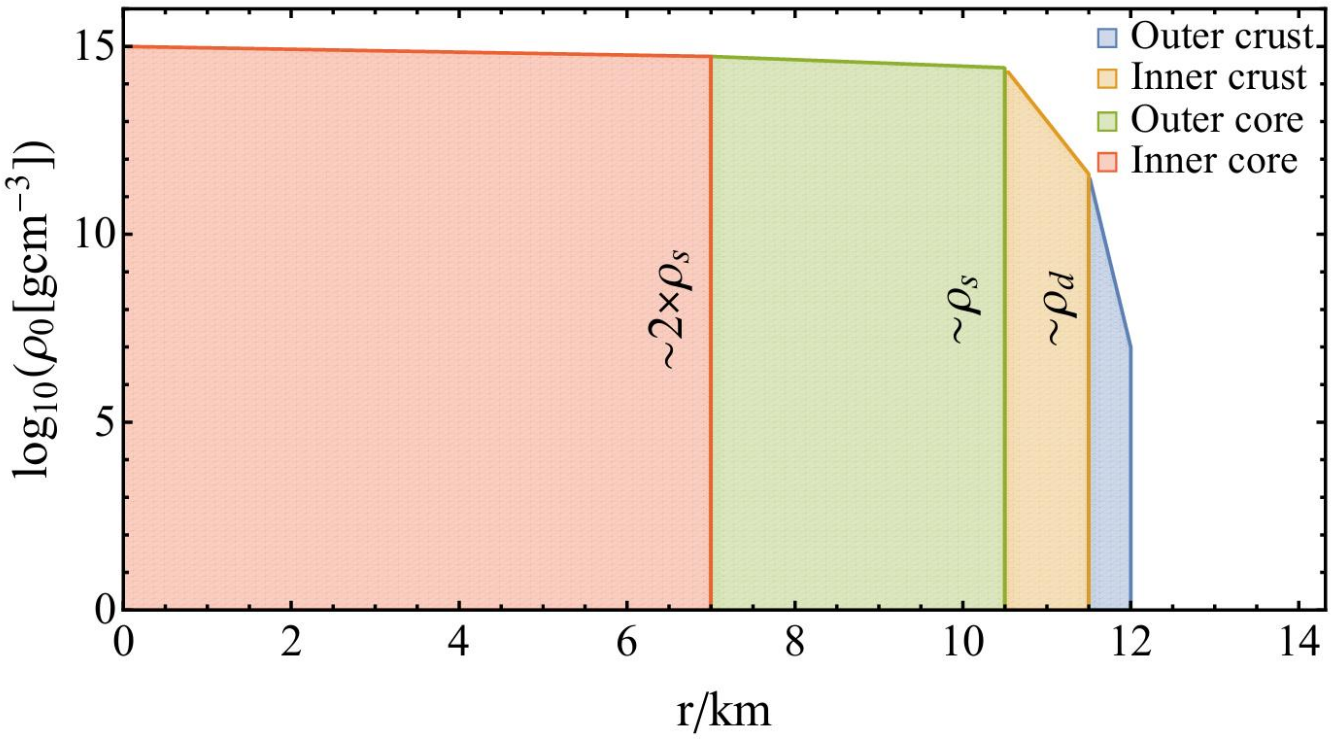
- Lattice of neutron-rich nuclei together with superfluid neutron gas and electron gas.

Outer Core:

- A homogeneous fluids layer, npeμ-matter.

Inner Core:

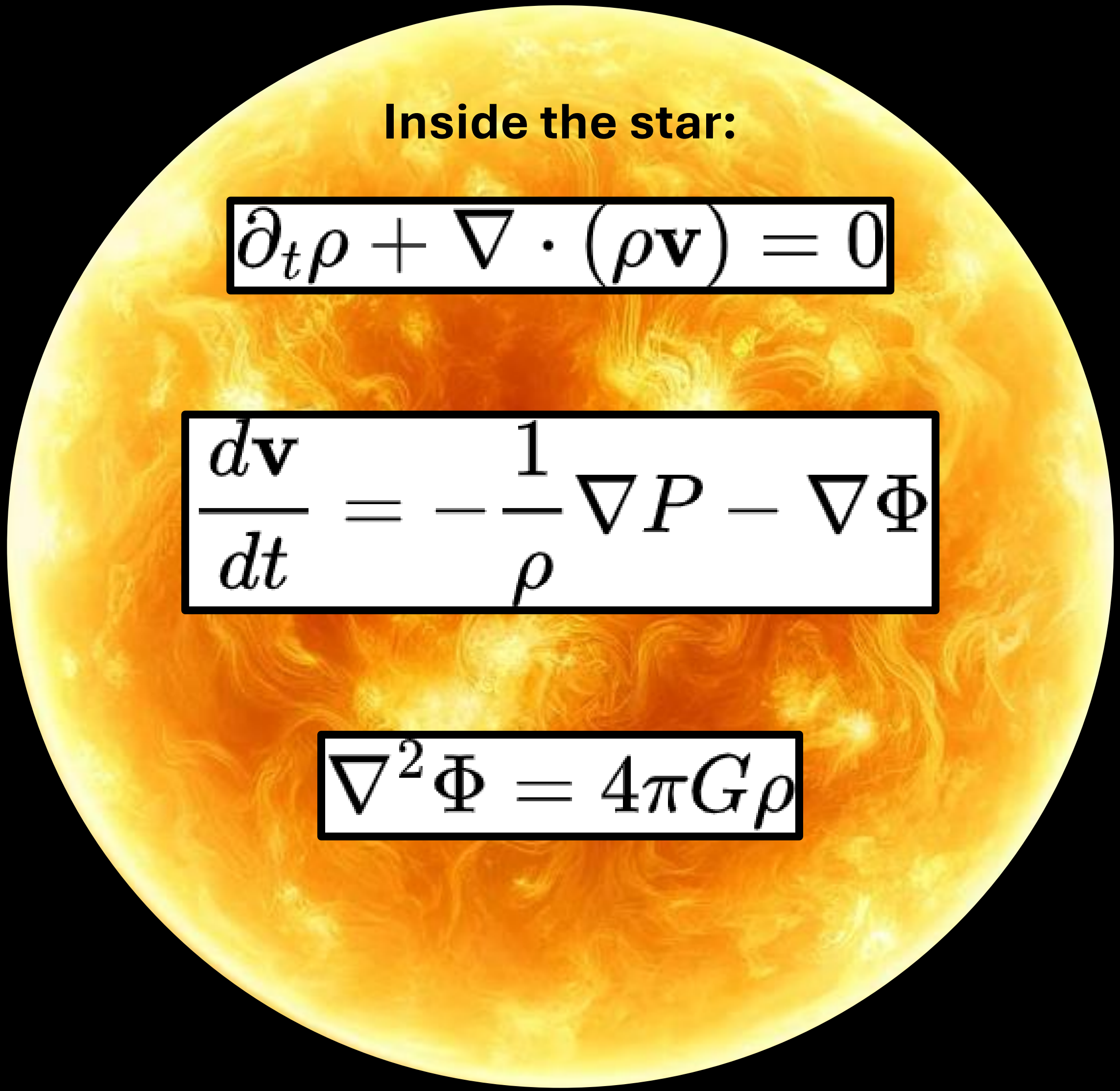
- Big questions here: deconfined quark matter, hyperons, Bose-Einstein meson condensates...



Quite complicated to model!

Neutron star EoS is one of the **main open problems** in astrophysics!

Self-gravitating fluids



Inside the star:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P - \nabla \Phi$$

$$\nabla^2 \Phi = 4\pi G \rho$$

Outside the star:

$$\rho = 0$$

$$\nabla^2 \Phi = 0$$

Tools to build a star!

The extension to GR was done in 1939.

[Clip of *Oppenheimer*, 2023]

Tools to build a star!

1) Our set of equations (from GR):

- Einstein's equations;
- Stress-energy tensor conservation.

2) Specify the form of metric ;

- e.g. spherically symmetric;

3) Some form for the stress-energy tensor:

- Perfect-fluid;

$$m' = 4\pi r^2 \rho$$
$$\phi' = \frac{m + 4\pi r^3 P}{r(r-2m)}$$
$$P' = -(\rho + P)\,\phi'$$

Tolman-Oppenheimer-Volkoff (**TOV**)Equations

FEBRUARY 15, 1939 PHYSICAL REVIEW VOLUME 55

On Massive Neutron Cores

J. R. Oppenheimer and G. M. Volkoff
Department of Physics, University of California, Berkeley, California
(Received January 3, 1939)

It has been suggested that, when the pressure within stellar matter becomes high enough, a new phase consisting of neutrons will be formed. In this paper we study the gravitational equilibrium of masses of neutrons, using the equation of state for a cold Fermi gas, and general relativity. For masses under $\frac{1}{3}\odot$ only one equilibrium solution exists, which is approximately described by the nonrelativistic Fermi equation of state and Newtonian gravitational theory. For masses $\frac{1}{3}\odot < m < \frac{3}{4}\odot$ two solutions exist, one stable and quasi-Newtonian, one more condensed, and unstable. For masses greater than $\frac{3}{4}\odot$ there are no static equilibrium solutions. These results are qualitatively confirmed by comparison with suitably chosen special cases of the analytic solutions recently discovered by Tolman. A discussion of the probable effect of deviations from the Fermi equation of state suggests that actual stellar matter after the exhaustion of thermonuclear sources of energy will, if massive enough, contract indefinitely, although more and more slowly, never reaching true equilibrium.

RELATIVITY THERMODYNAMICS AND COSMOLOGY

BY
RICHARD C. TOLMAN
PROFESSOR OF PHYSICAL CHEMISTRY AND MATHEMATICAL PHYSICS AT THE CALIFORNIA INSTITUTE OF TECHNOLOGY

1) The final ingredient: *Equation of State!*

Specifies the **microphysics** of the body. In general, can be quite complex.

$$\rho = \rho(n, s)$$

Simplification: The fluid is **adiabatic** and **isentropic.**

$$\rho = \rho(n)$$

$$P = P(n)$$

$\Rightarrow$

$$P = P(\rho)$$

Most familiar form!

Fluid-ball conjecture

Static and **asymptotically flat** fluid solutions are **spherically symmetric!**

Proved by [Massod-ul-Alam, 2007] for realistic case scenarios.

Equation of State

In general: No analytical solution.

Special case: Constant density star.

$$P = \rho_0 \frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}}$$

What happens to our star when we increase the central pressure?

$$M/R \to 4/9$$

Buchdahl's Bound:

Under some set of assumptions, the compactness of a self-gravitating object must be bounded by:

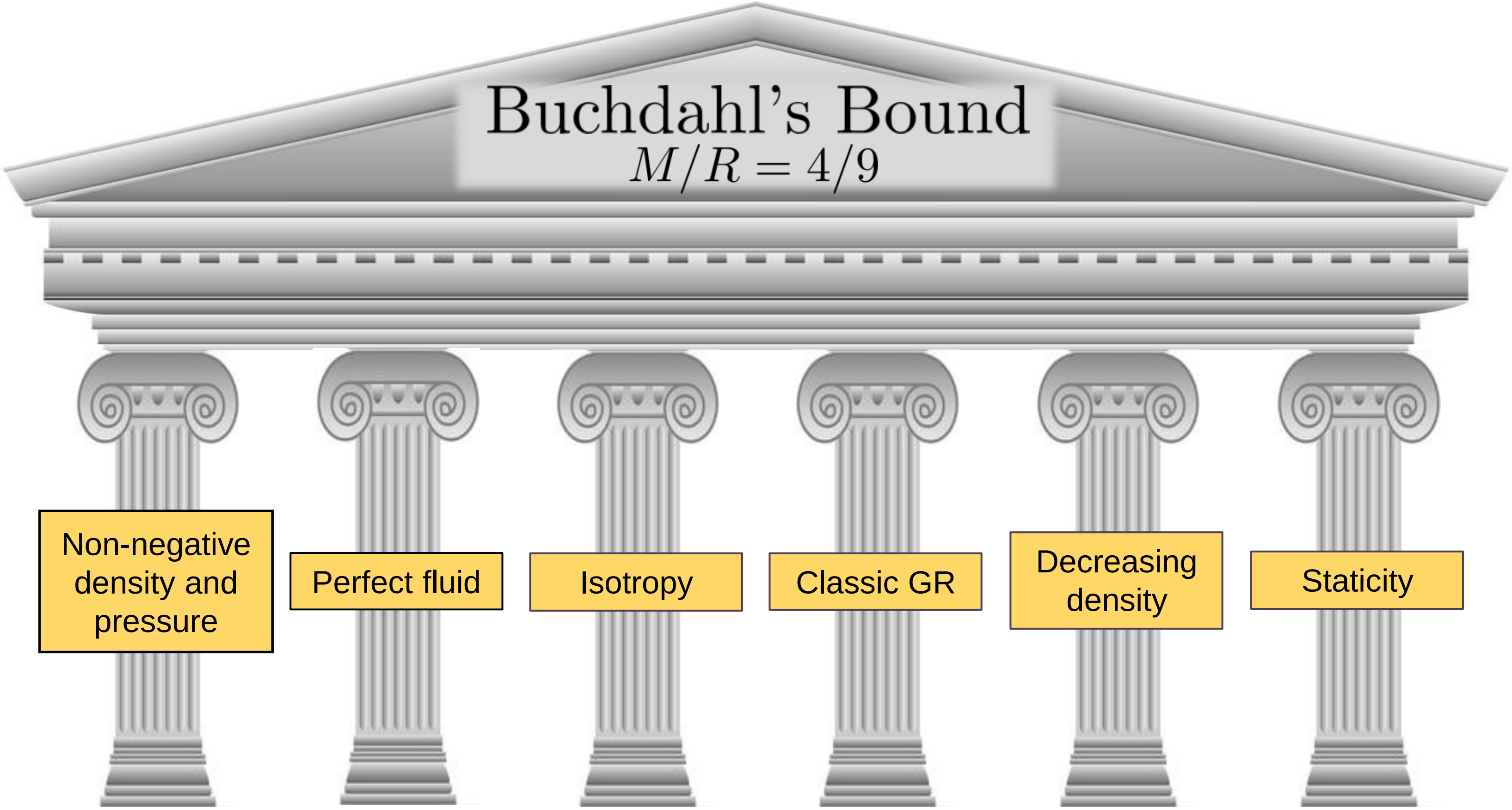
$$M/R < 4/9$$

PHYSICAL REVIEW VOLUME 116, NUMBER 4

General Relativistic Fluid Spheres

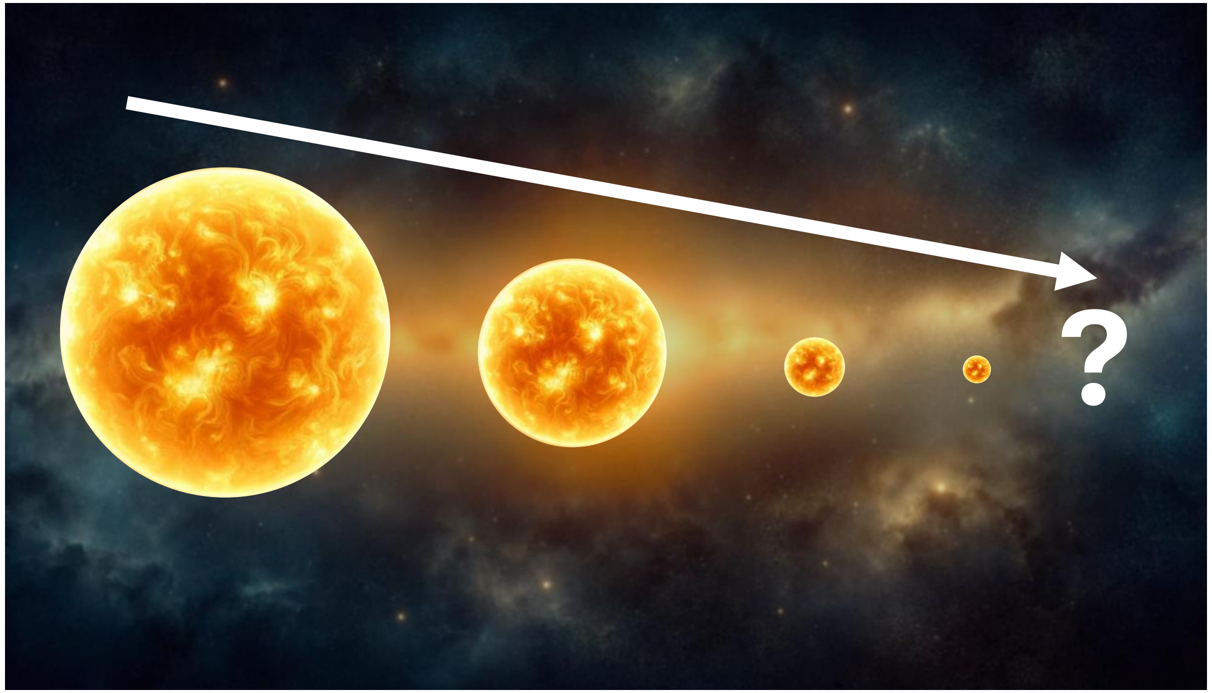
H. A. BUCHDAHL*
Institute for Advanced Study, Princeton, New Jersey
(Received June 16, 1959)

In Part I of this paper certain well known results concerning the Schwarzschild interi generalized to more general static fluid spheres in the form of inequalities comparing the bou g_{44} with certain expressions involving only the mass concentration and the ratio of the central to the central pressure. A minimal theorem appropriate to the relativistic domain is derived pressure, corresponding to a well-known classical result. Inequalities involving the proper potential energy are also considered, as is the introduction of the physical radius in place of radius. A singularity-free elementary algebraic solution of the field equations is presented ar obtained from it compared with the limits prescribed by some of the inequalities. In Part I given to the question whether the total amount of radiation emitted during the symmetrica contraction of an amount of matter whose initial energy, at complete dispersion, is W_0 can e



Maximum Compactness of Stars

Let's go back to Buchdahl. We know that Buchdahl is a limit, but does it make **physical sense**?



Incompressible fluid = Infinite Sound Speed! Not very realistic.

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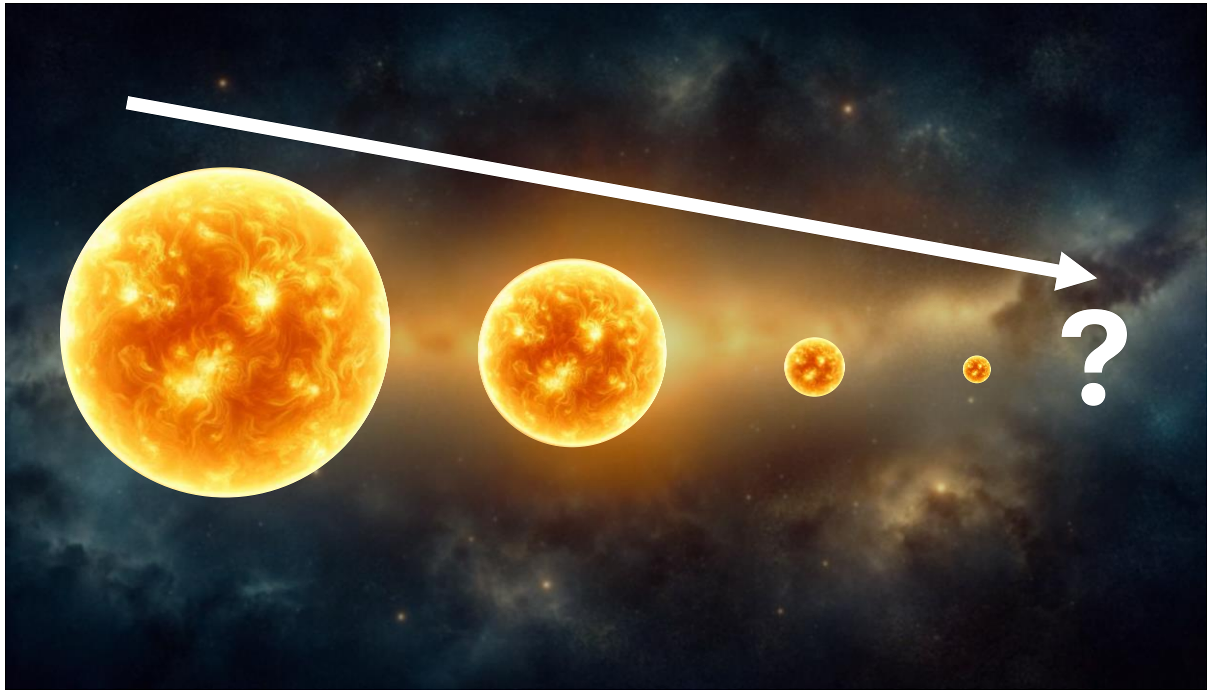

Incompressible fluid = Infinite Sound Speed! Not very realistic.

What is the **highest compactness** of a **physically viable** compact object?

[My very real ultracompact backpack in Marajó]

Integration of TOV equations

When you cannot do it analytically – Integrate numerically!

1. Pick a value of the central density. The equation of state gives the central pressure.

2. Integrate the system from r=0 outwards. EOS is used at each point to calculate the density

3. When to stop calculation?

   - When Pressure is zero, we have found the radius of the star!

4. What to do with the initial value of the potential?

Some analytical EoS

- **Constant Density:** Checked! Leads to Buchdahl limit.

Some analytical EoS

- **Constant Density:** Checked! Leads to Buchdahl limit.
- **Constant adiabatic index:** Two families of EOS: (Tooper, 1965)

$$P = K\varrho^{\gamma}$$

1) Polytropes (Tooper, 1965);

$$\rho = CK^{1/\gamma}\varrho + \frac{K}{\gamma - 1}\varrho^{\gamma}$$

no bounded solutions for n>5

2) **Linear** constant sound speed (Bondi, 1964):

$$\rho = \frac{K}{\gamma - 1}\varrho^{\gamma} \Rightarrow P = (\gamma - 1)\rho$$

Scale-invariant, but **no bounded** solutions!

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- **Affine** constant sound speed:

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$$\rho = \frac{\gamma - 1}{\gamma}\rho_0 + \frac{K}{\gamma - 1}\varrho^{\gamma}$$

Always bounded. This class includes *Christodoulou's hard phase material* and MIT bag model (quark stars).

Maximum Compactness of Stars

With **constant sound speed EoS** we can look for bounds on **viable** stars!

Maximum Compactness of Stars

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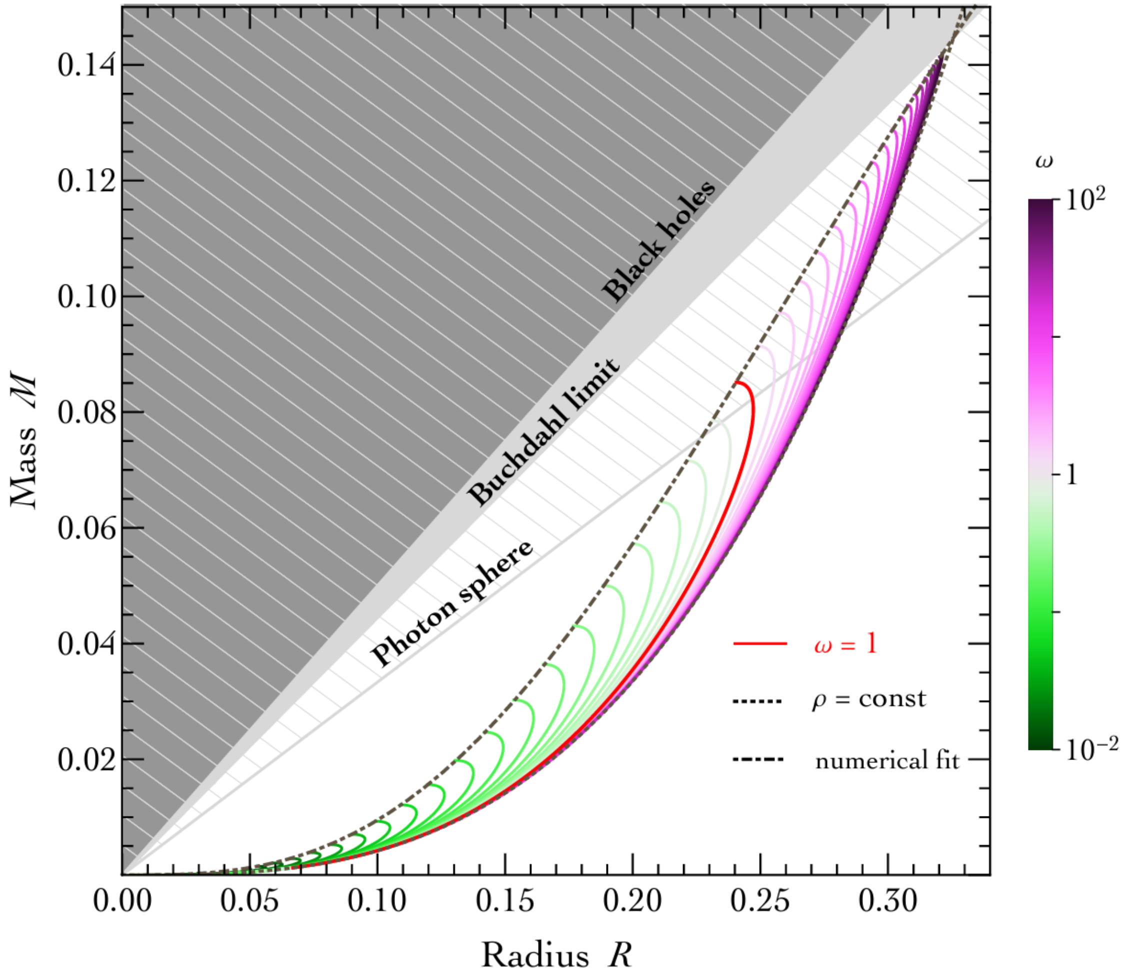
Black Hole: $C = 0.5$

Buchdahl Bound: $C = 4/9$

Causal Buchdahl Bound: $C = 0.364$

Causal Buchdahl bound + Radial Stability:

$$C = 0.354$$

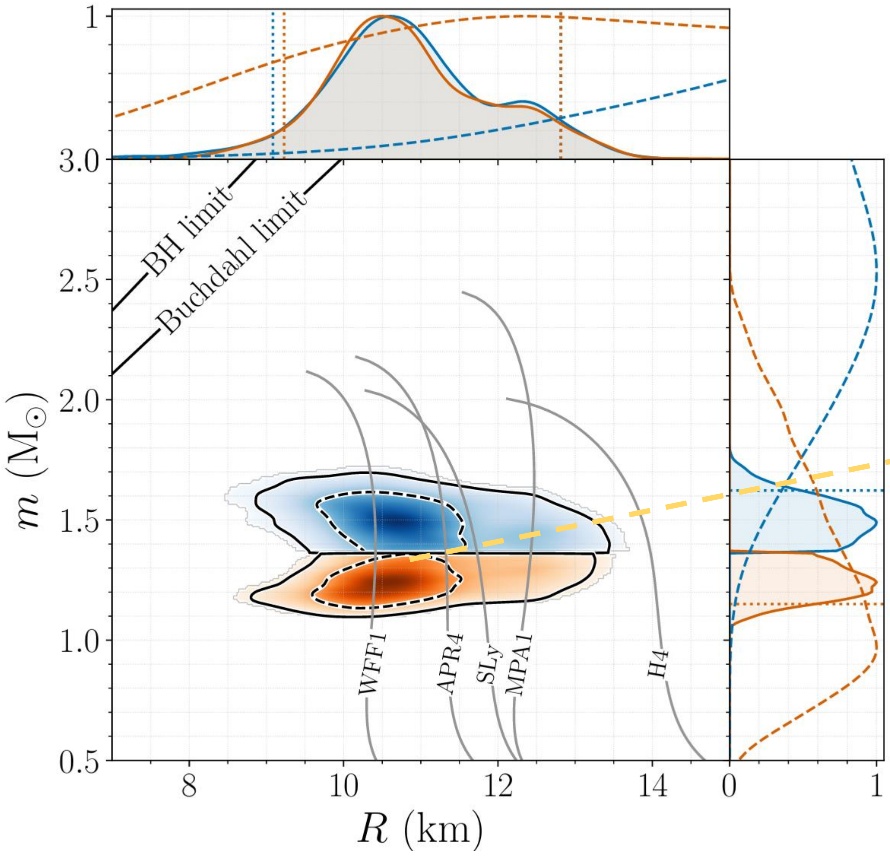


[Urbano & Vermaee, 1810.07137]

Realistic approximations of NSs

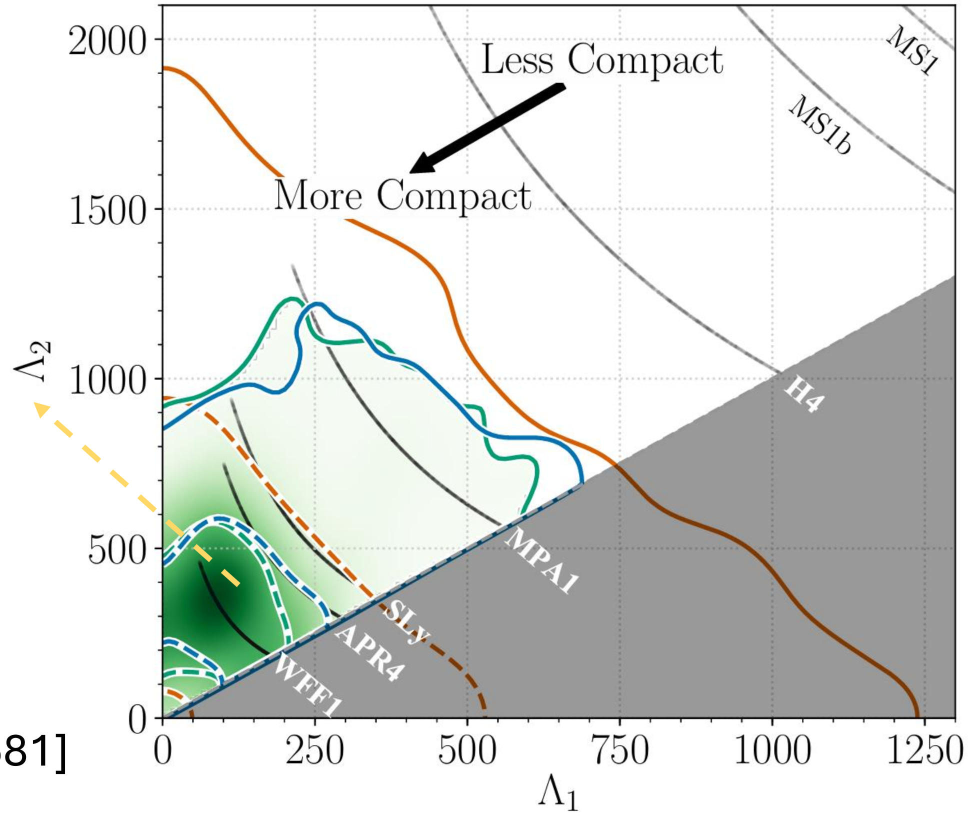
Tabulated EOS for Neutron Stars:

Construct EOS tables based on nuclear physics models. (APR4, Sly, MPA, H4, MS1, etc..)



"Soft" EoS are preferred over "Stiff" EoS

[Ligo/Virgo, 1805.11581]



Realistic approximations of NSs

Piecewise Polytrope

Different neutron star layers are approximated by different polytropes. (3 is good enough).

- Crust: Degenerate gas of relativistic electrons. *(see Chapter 2, Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects,* Shapiro & Teukolsky)

$$\gamma < 4/3$$

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- Middle: Degenerate gas of non-relativistic neutrons.

$$\gamma = 5/3$$

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$$\gamma < 4/3$$

- Middle: Degenerate gas of non-relativistic neutrons.

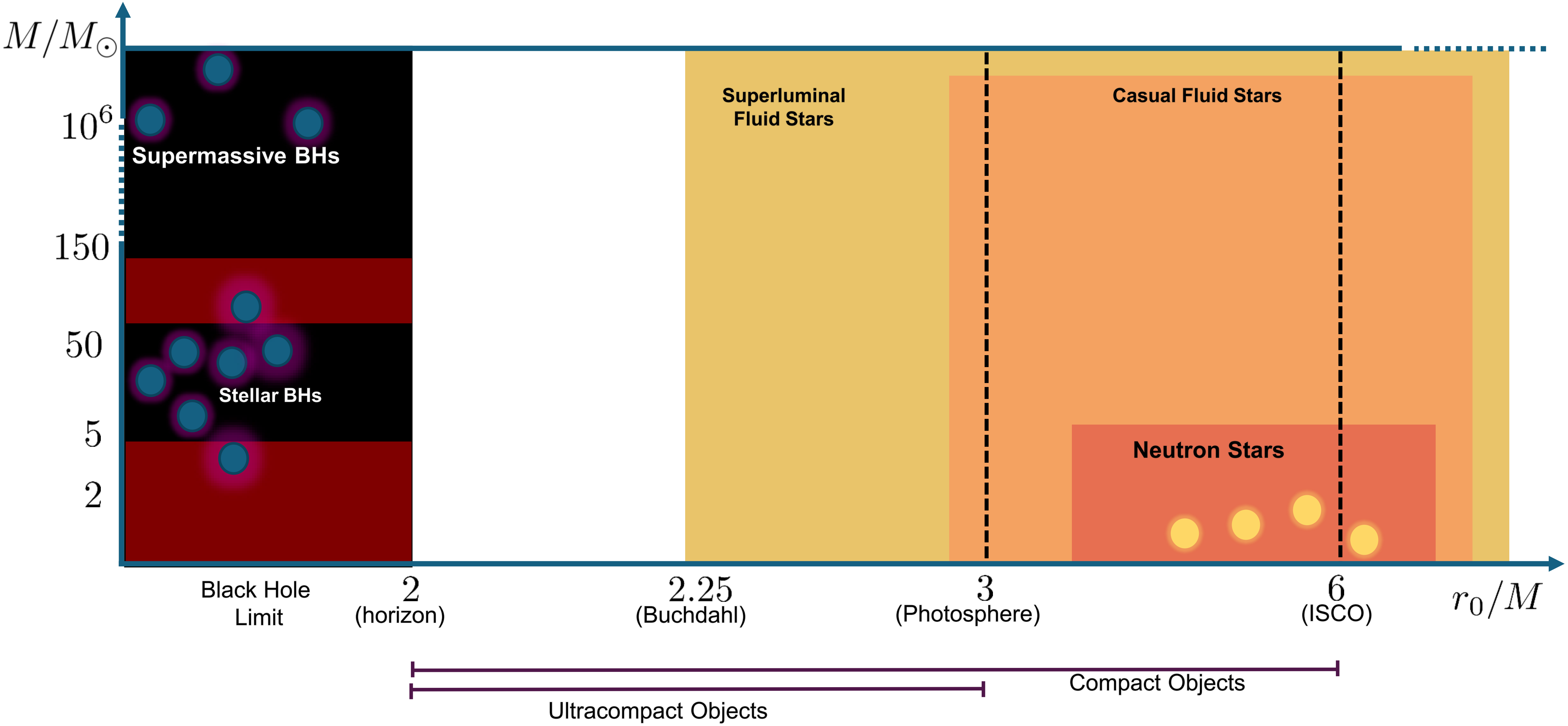
$$\gamma = 5/3$$

- Core: Gas of ultra-relativistic quarks/fermions.

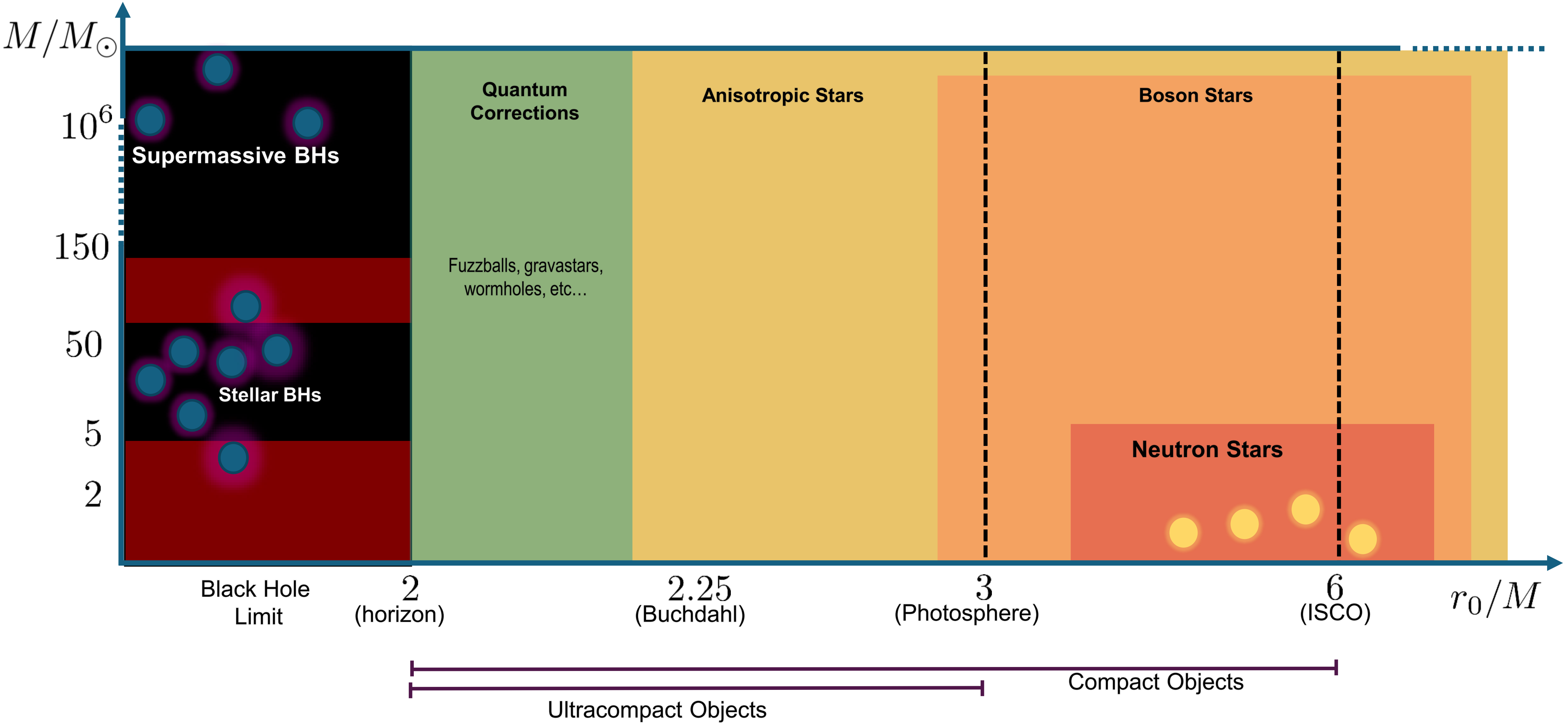
$$\gamma = 1$$

All pieces are “Soft” EoS.

Maximum Compactness of Stars



A Zoo of Compact Objects



Exotic Universe

Why do we care about this?

Motivation #1: “The skeptical”.

Black holes are also “exotic”. Singularity at the center and a horizon as a surface.

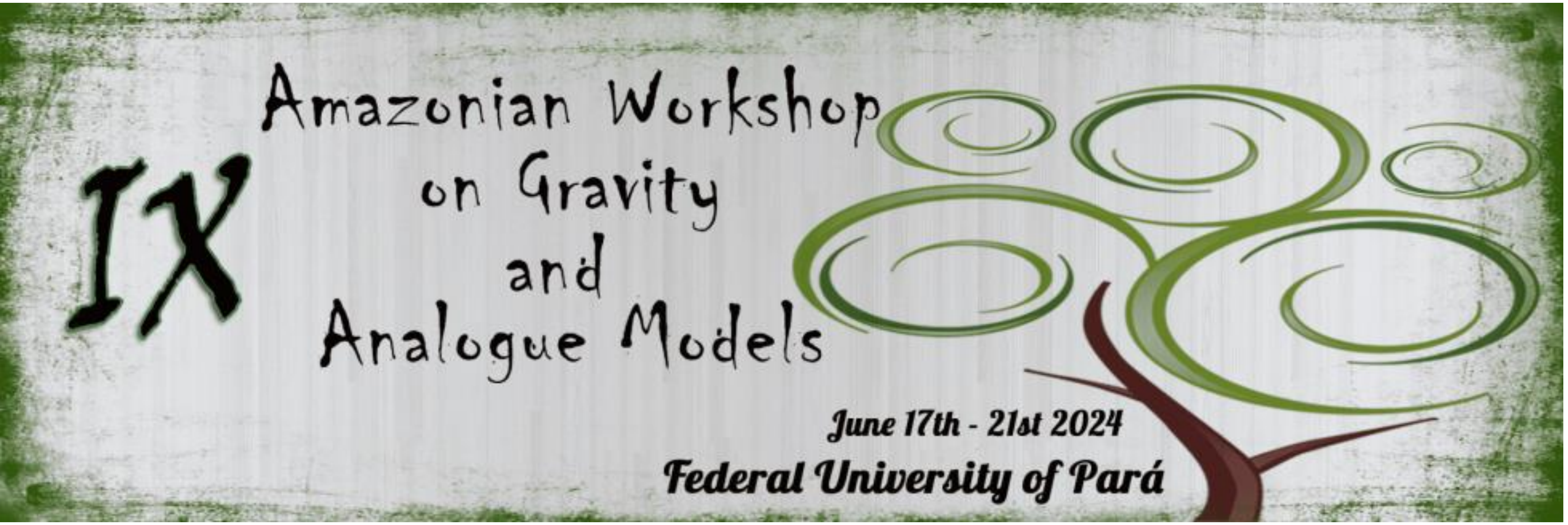
Motivation #2: “The idealist”

Black holes and Neutron stars may be just 2 species in a larger *Zoo of Compact Objects*.

Motivation #3: “The pragmatic”

Constraining everything else would help us validate the black hole model.

Compact Objects and How to Model Them
Part I



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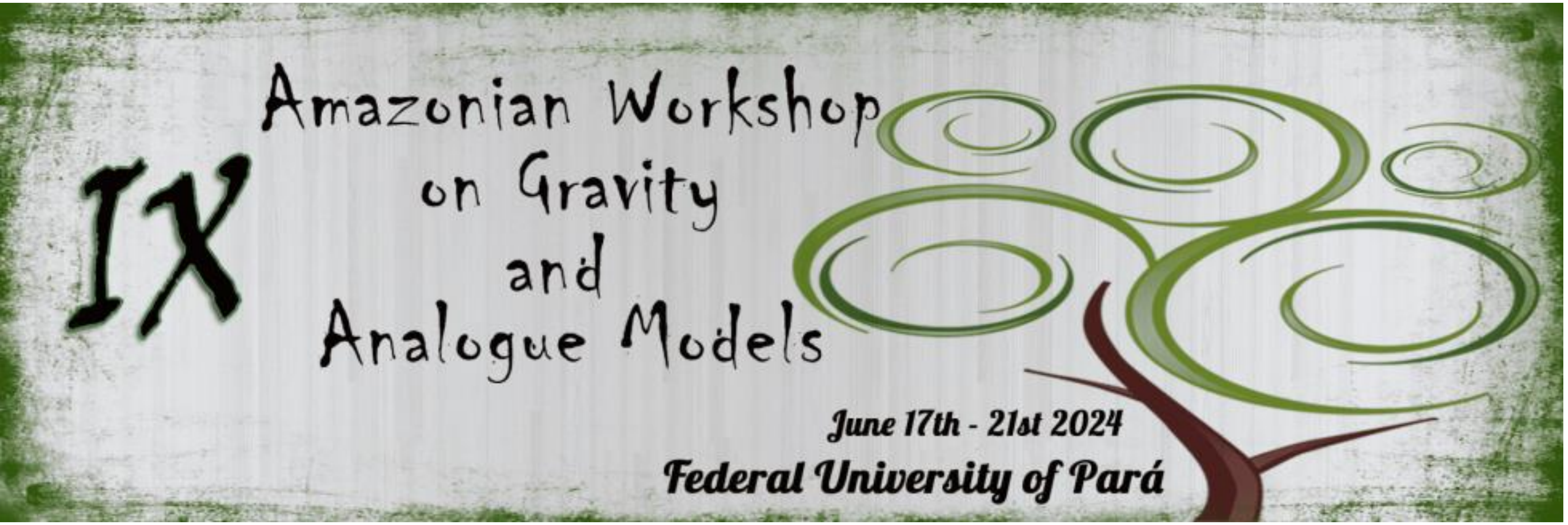

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Last lecture:

Compact Objects and Perfect Fluids:

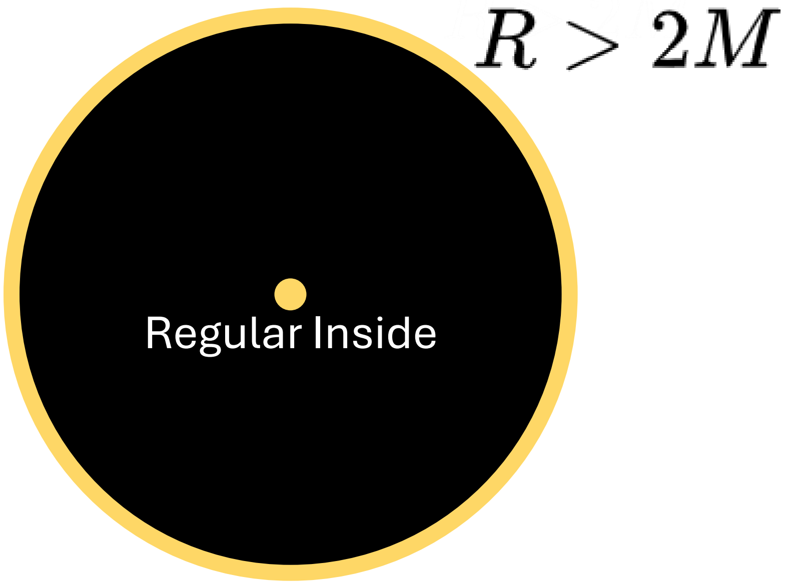
- What are compact objects?
- Self-gravitating fluids
- Equation of State
- Buchdahl limit

Our ECO Wishlist

- If an ECO represents a BH alternative it should have less problems than BHs have.

- If they form in Nature, we want:

  1. Horizonless and Singularity free!



Our ECO Wishlist

- If an ECO represents a BH alternative it should have less problems than BHs have.

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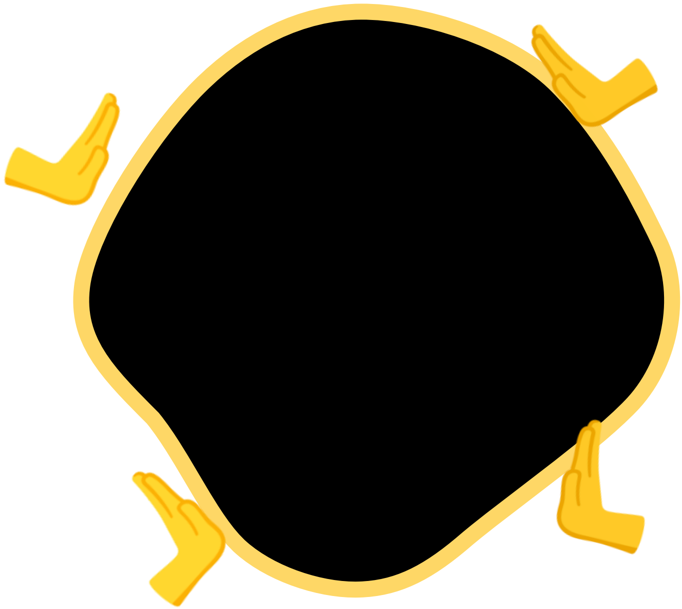
  2. Stable



How long can it live?

Our ECO Wishlist

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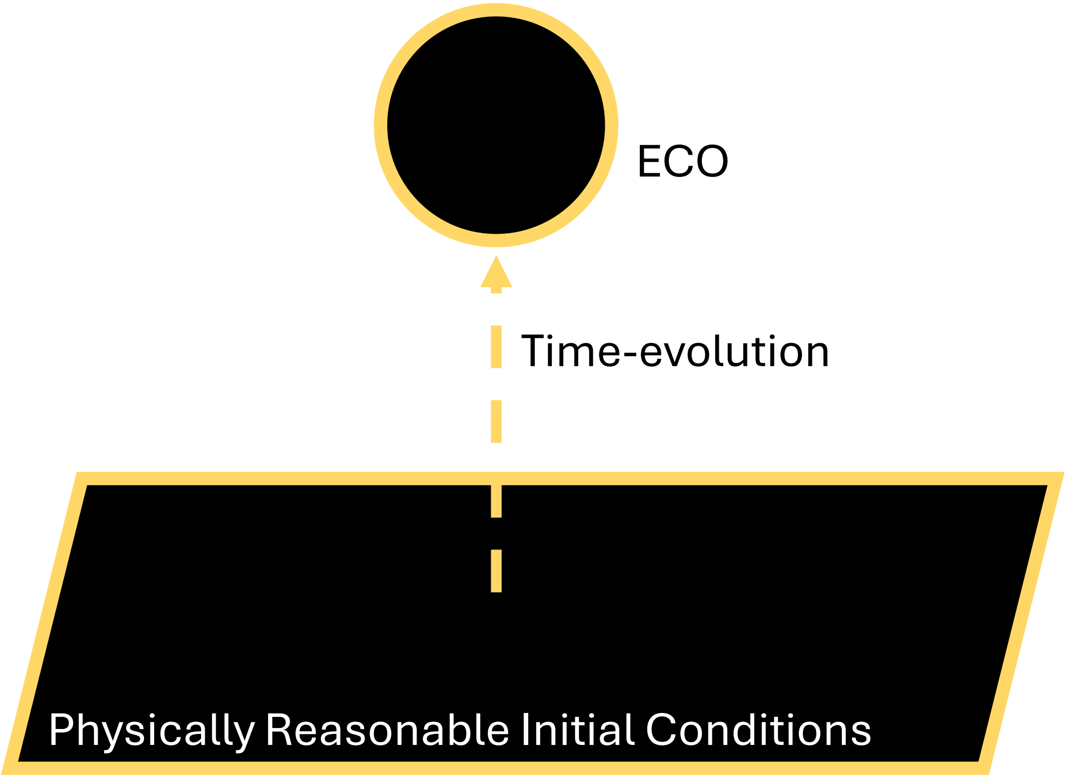
  3. Formation mechanism



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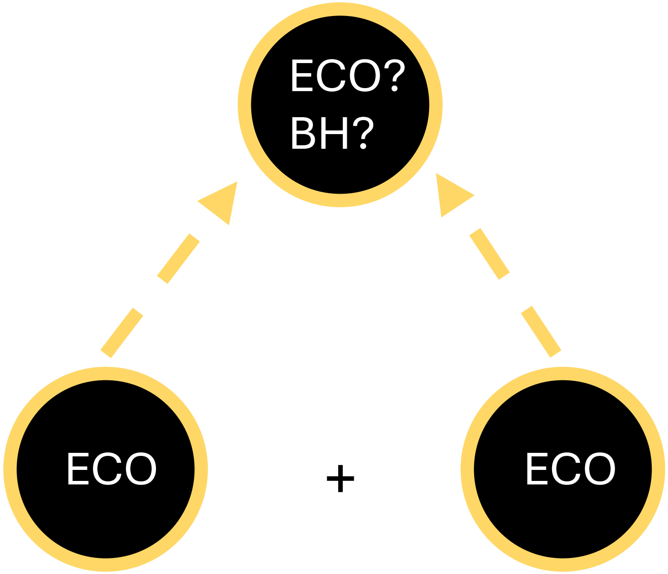
  3. Formation mechanism
  4. Well understood dynamics



2 Approaches

Parametrized
ECO
Model

Pick your favourite and study it!

OR

Build an ECO modelled with some general parameters

Compass to construct ECOs

Buchdahl's Bound:

Under some set of assumptions, the compactness of a self-gravitating object must be bounded by:

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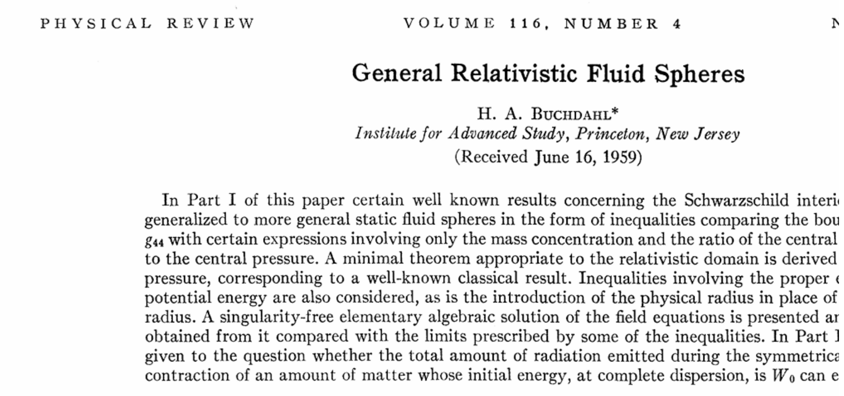
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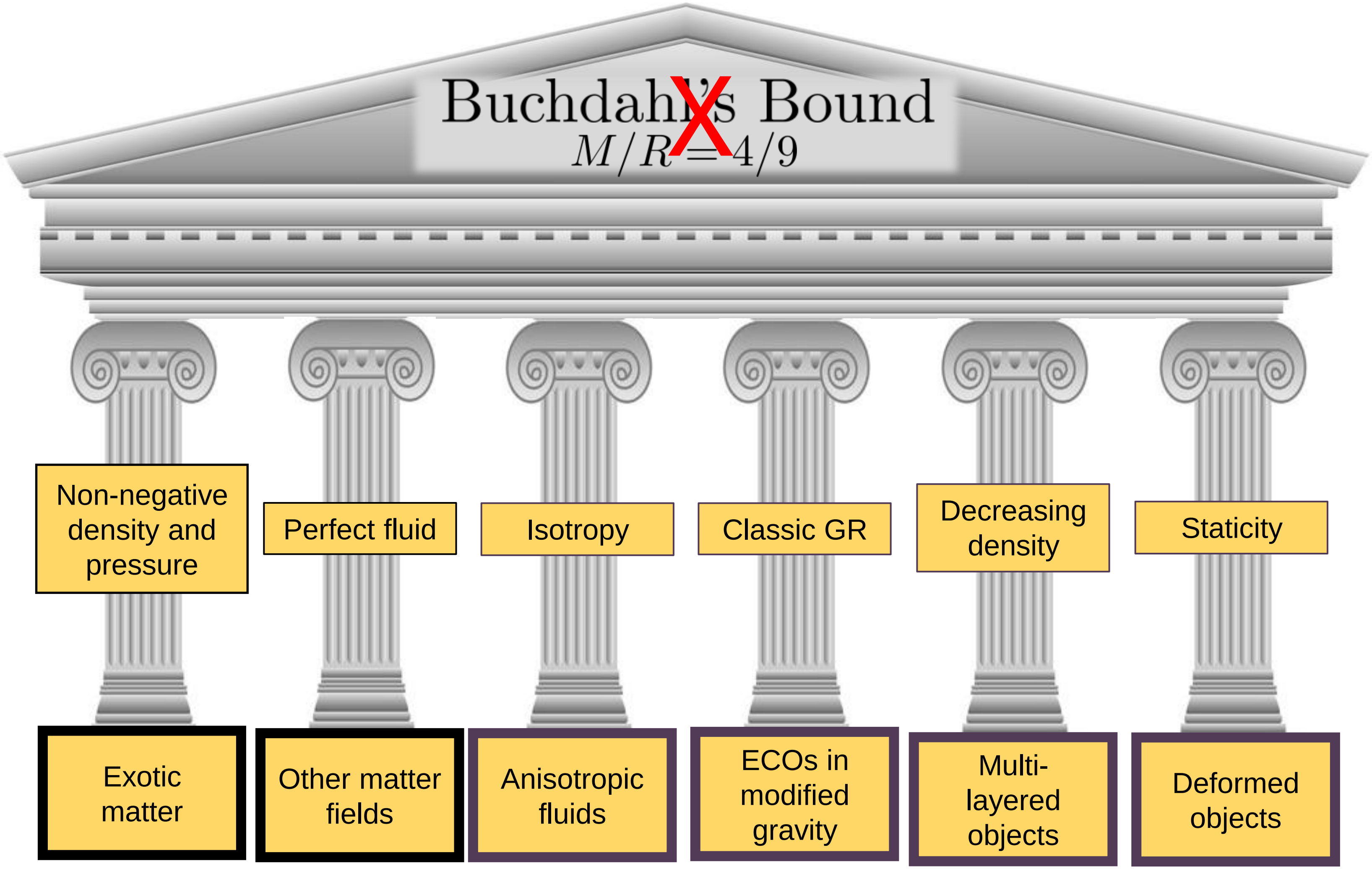

Exotic Compact Object Models

Case 1: The "vanilla" wormhole case.

External Vacuum Solution
(typically, Schwarzschild or Kerr)

$$ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\Omega^2$$

Interior Model

$$ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\Omega^2$$

Boundary conditions
on the surface

$R = r_0$

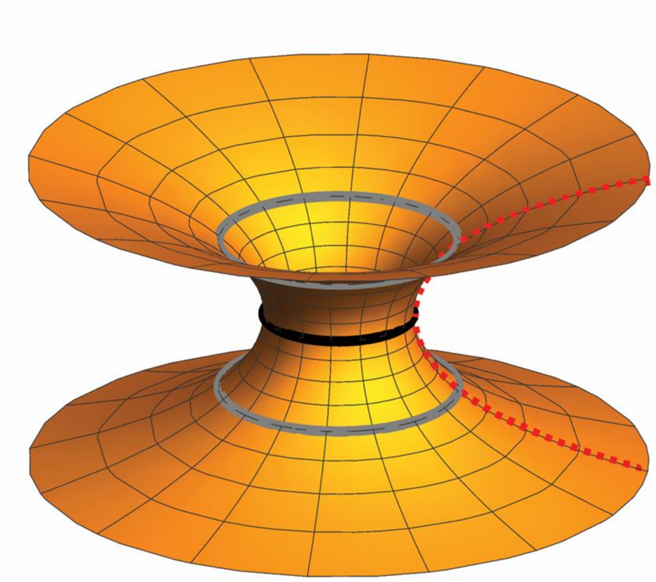

Fig. from [Cardoso, Franzin, Pani, 2016]

Exotic Compact Object Models

Case 2: The “gravastar” case. [Mazur, Mottola, 2001]

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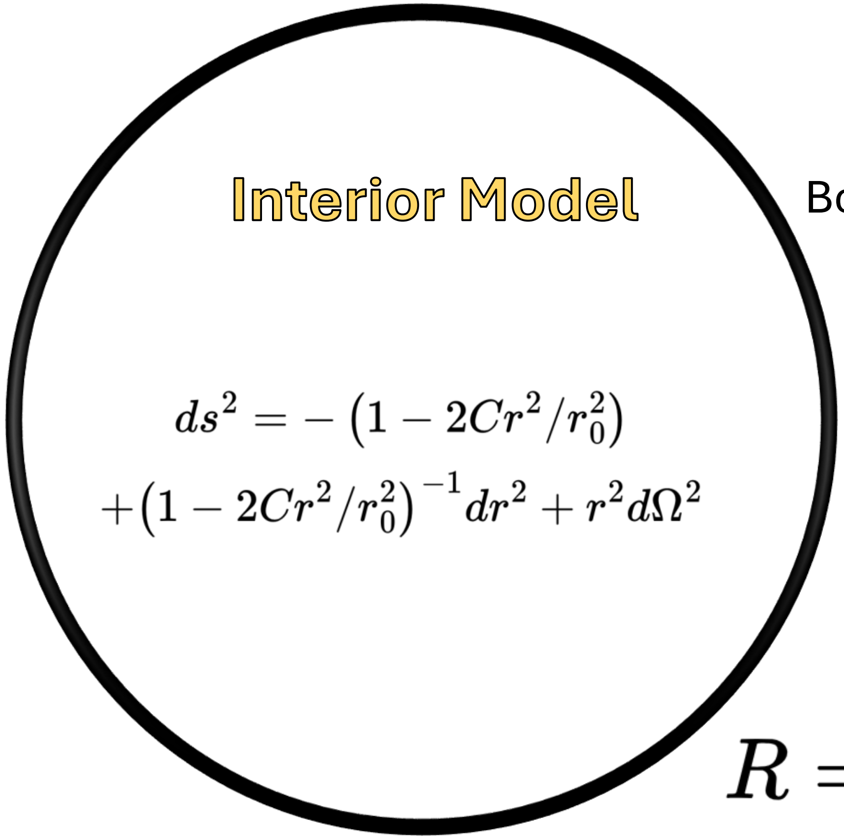

$$ds^2 = -\left(1 - 2Cr^2/r_0^2\right) + \left(1 - 2Cr^2/r_0^2\right)^{-1} dr^2 + r^2 d\Omega^2$$

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[Visser, Whiltshire, 2004]

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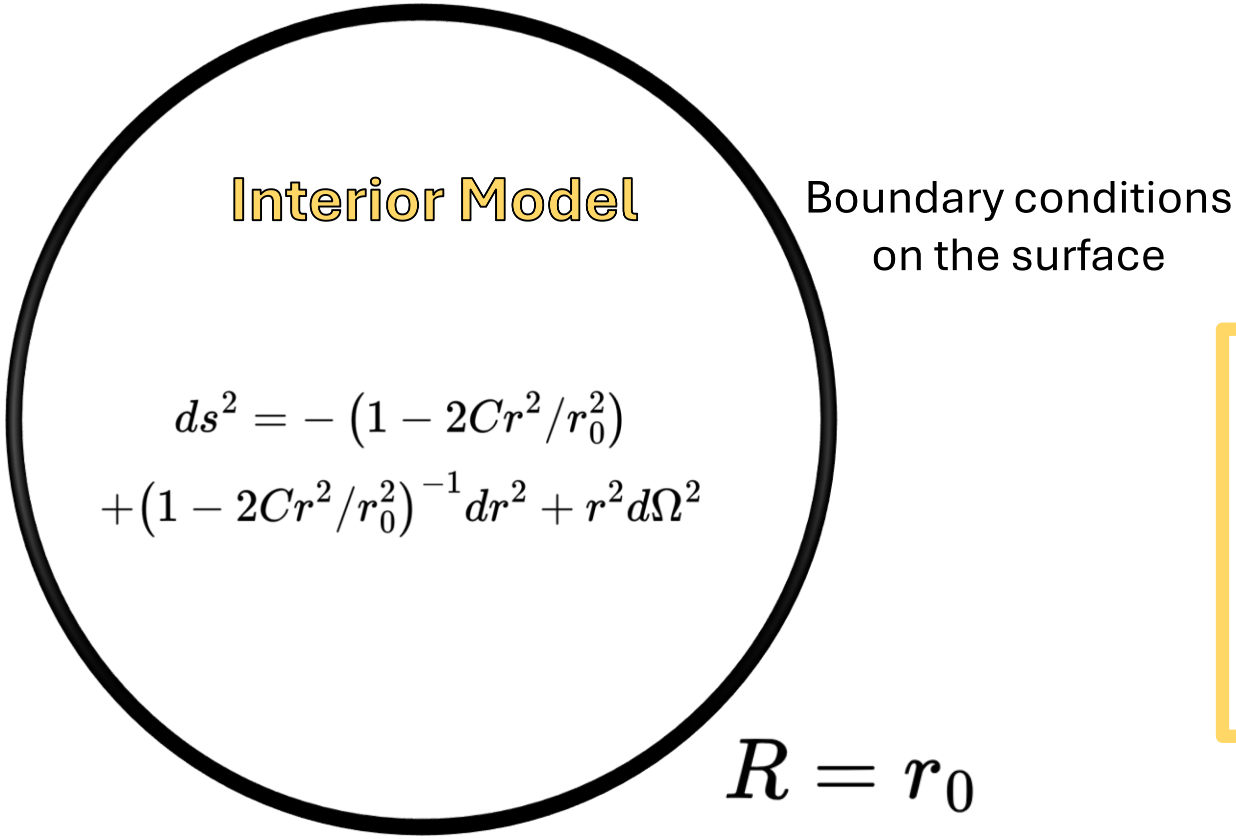
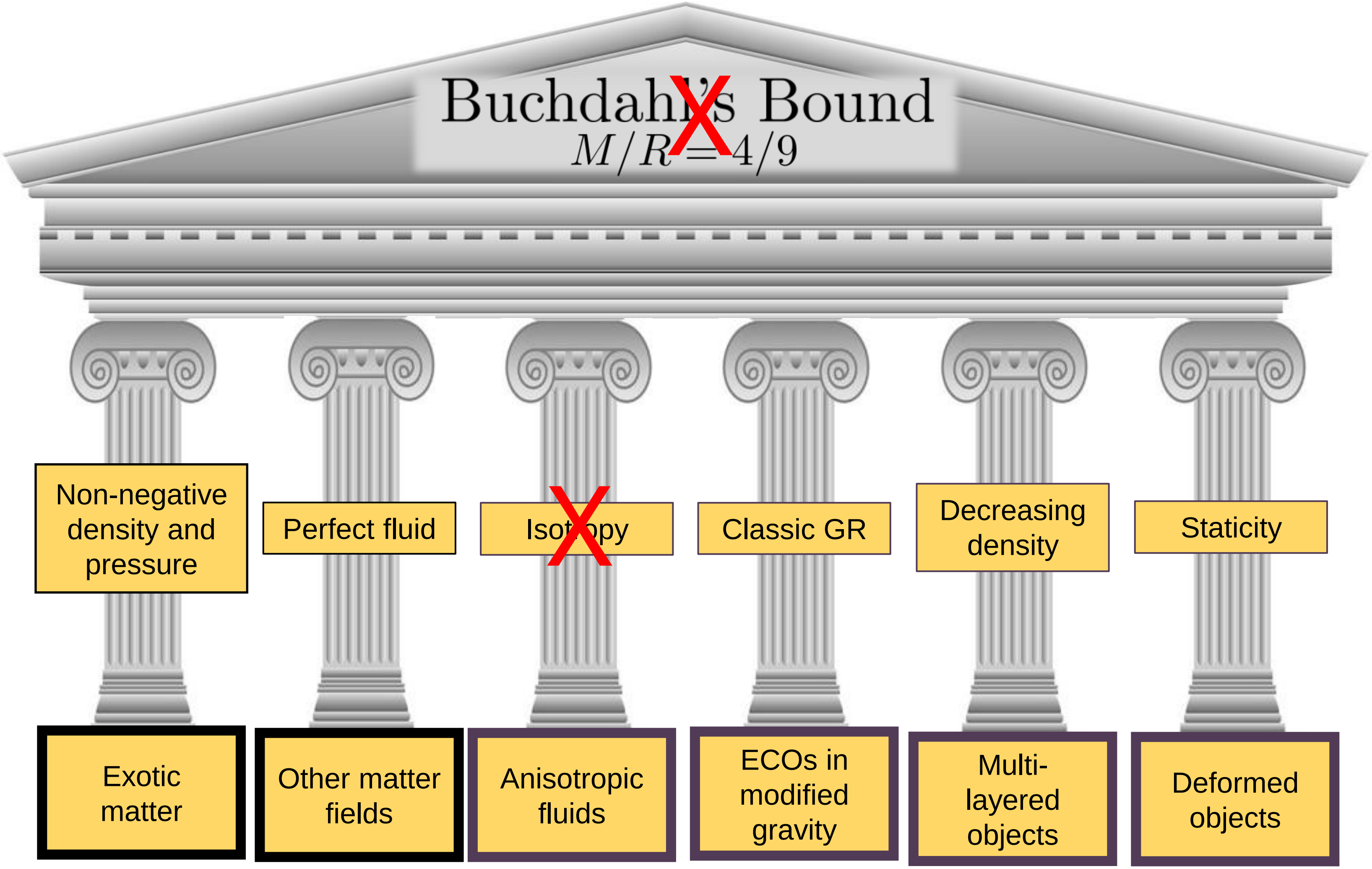


[Visser, Whiltshire, 2004]

This construction of ECOs is very forced.

- No dynamics or formation;
- Stability studies are complicated.

Anisotropic Stars



Anisotropic Stars

- The first discussion of anisotropy in the context of stars dates from [J. Jeans, 1922]
  - Context of “Kapteyn-spheroidal stars”.

- “Recently” the interest in anisotropic stars started with [Bowers & Liang, 1974].

- Several works in the past have explored the structure and properties of anisotropic stars.
  [Heintzmann & Hillebrand, 1975; Herrera, 2013; Biswas & Bose, 2019; etc.]

- **However**, anisotropic stars have some problems.

Anisotropic Stars

Stress-energy tensor of an anisotropic fluids

$$T^{\alpha}_{\ \beta} = \mathrm{diag}(-\mu, p_{||}, p_{\perp}, p_{\perp}),$$

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Solution is **singular** unless **anisotropy vanishes** at the centre!

Anisotropic Stars

The anisotropic mechanism must make the pressure isotropic at the center.

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$$P_r - P_t = C g_{rr} \left(\rho + P_r\right) \left(\rho + 3P_r\right) r^2$$

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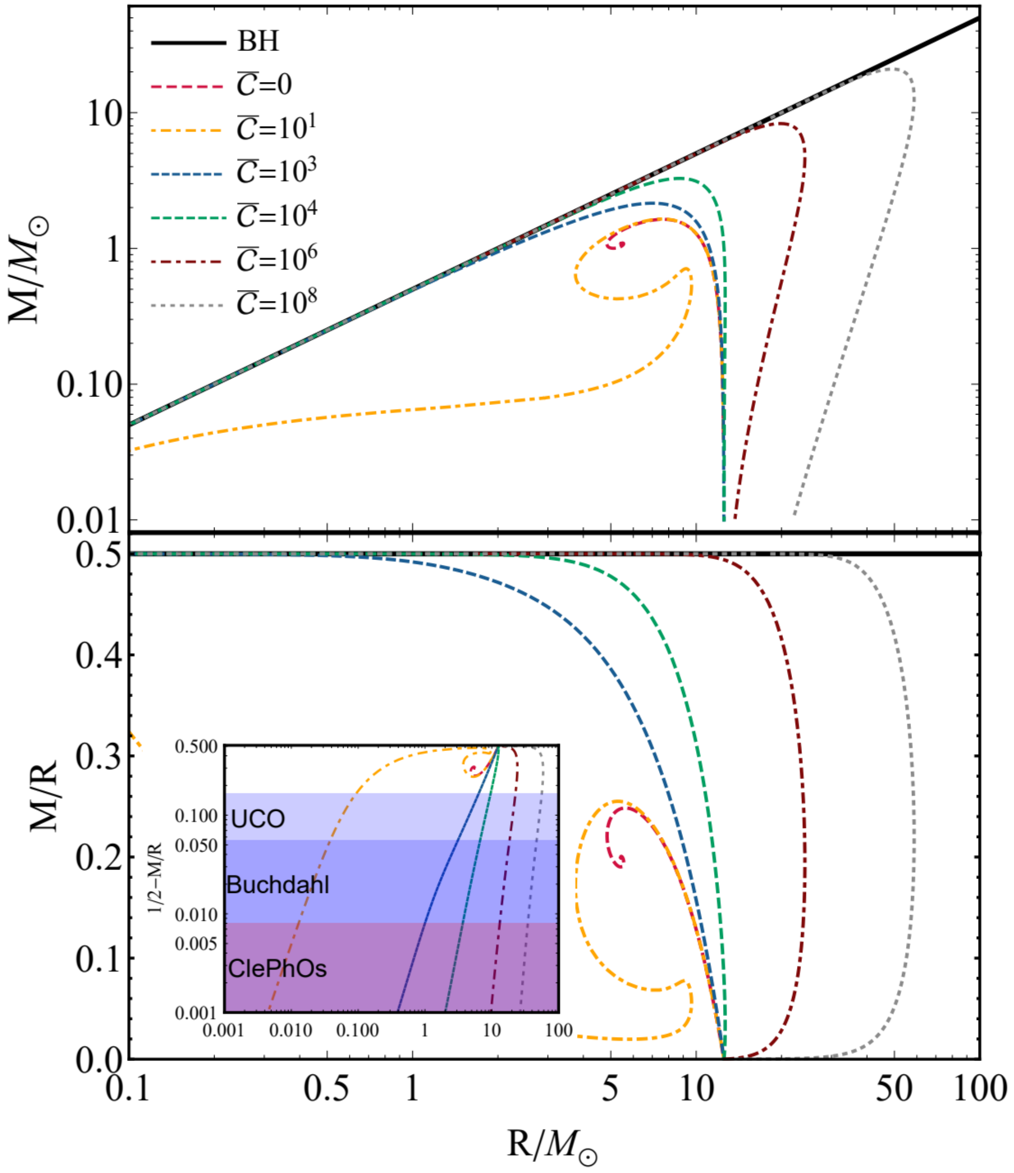
1. Extremely compact configurations! More compact and massive than isotropic fluid stars! Always approach Schwarzschild compactness.
2. Can exist in a wide range of mass!



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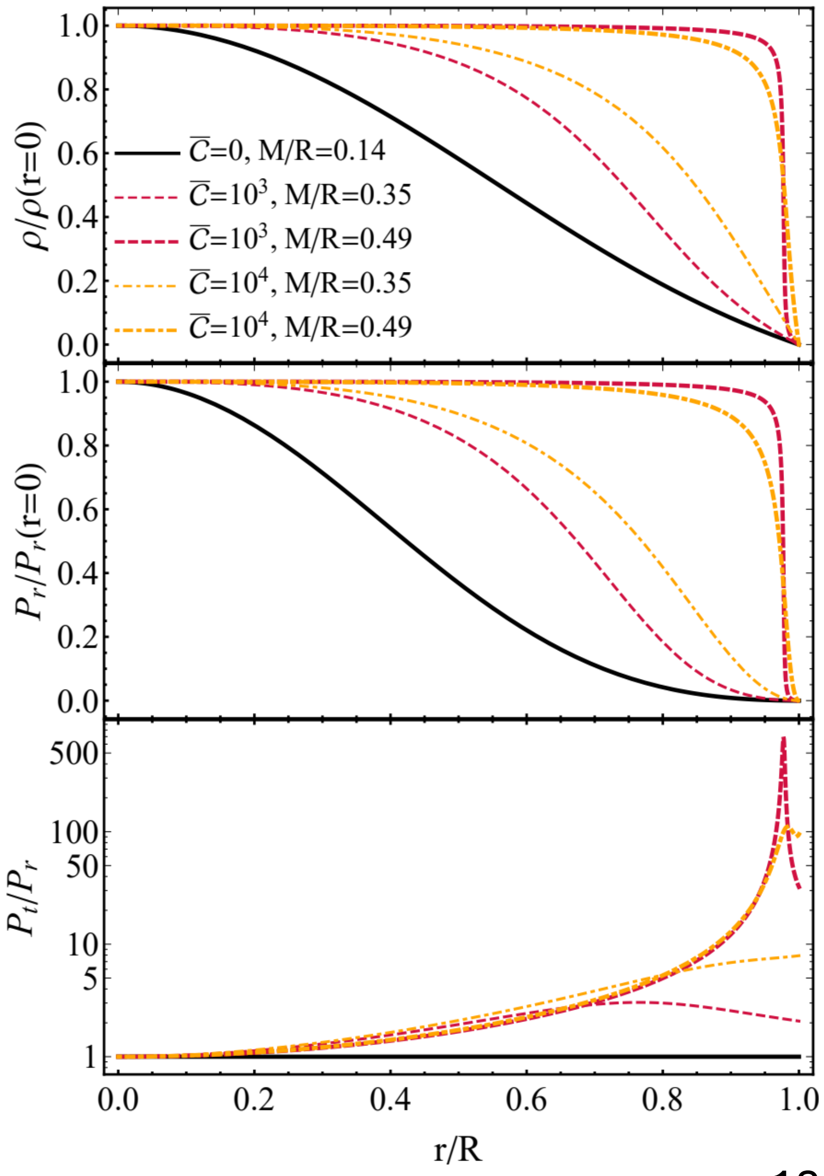

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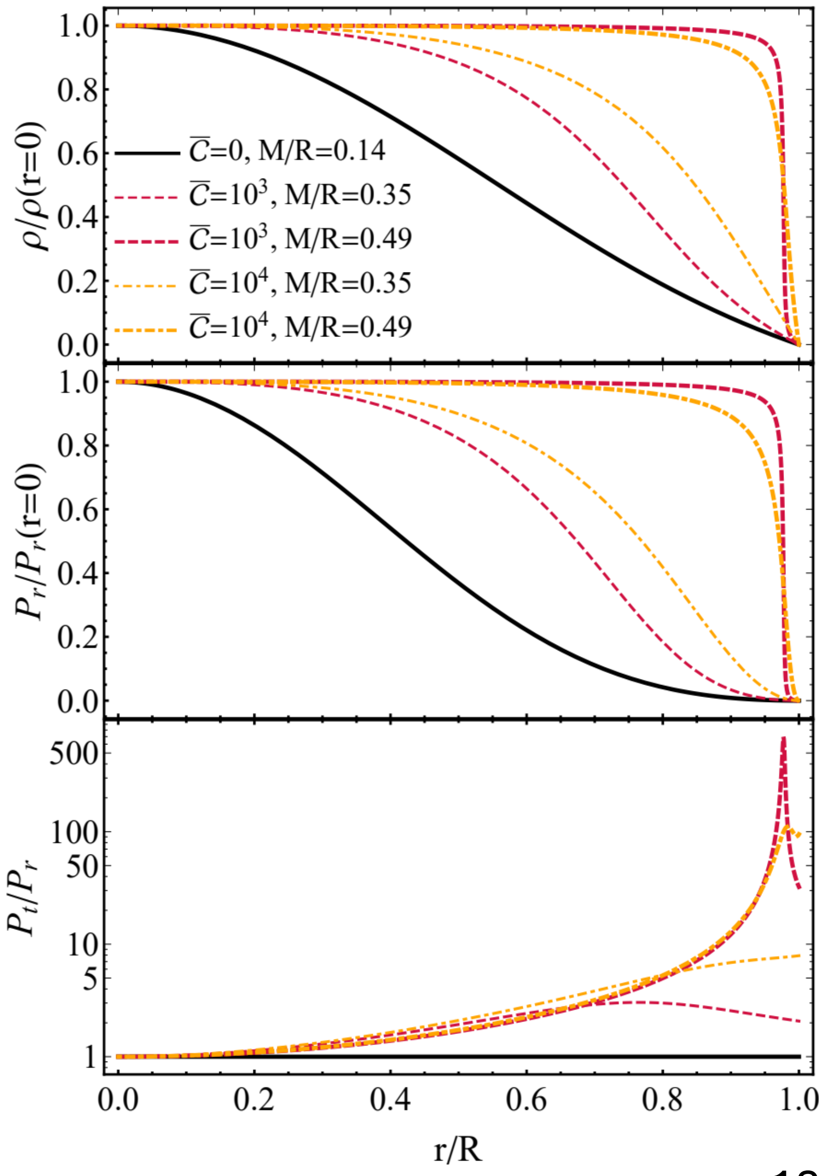

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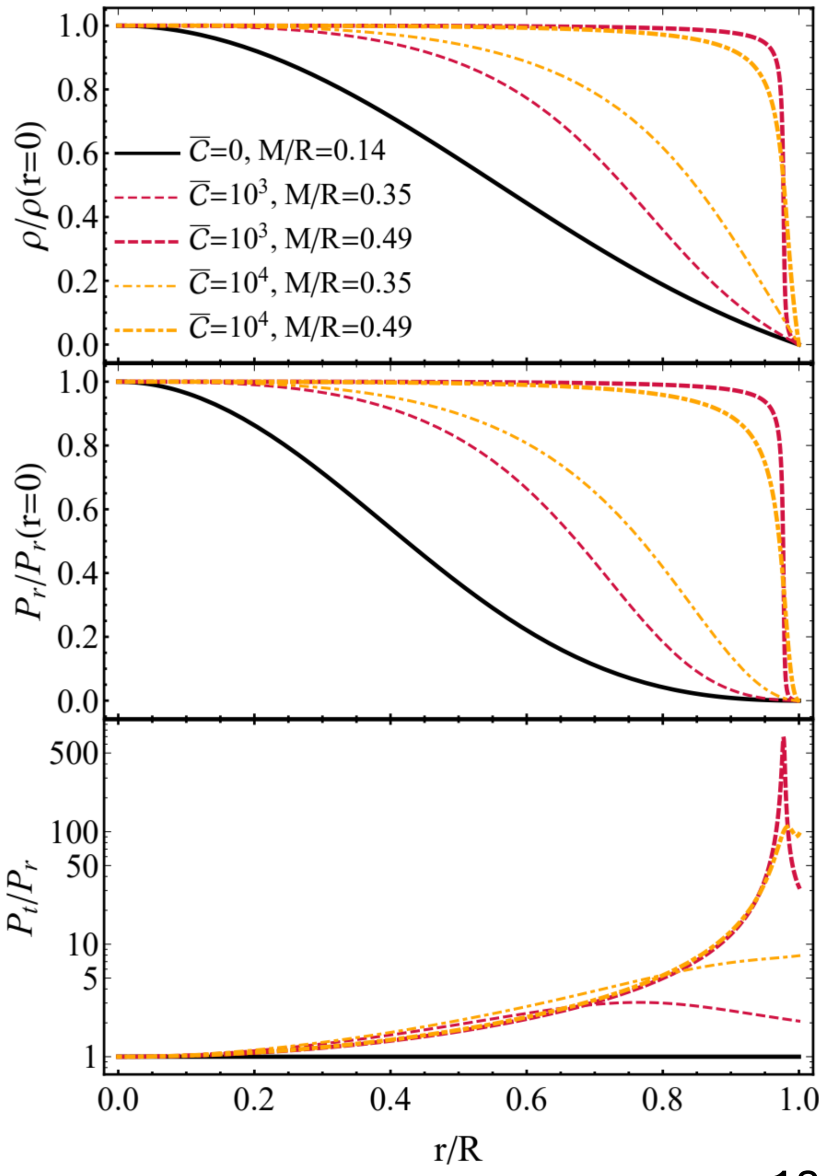
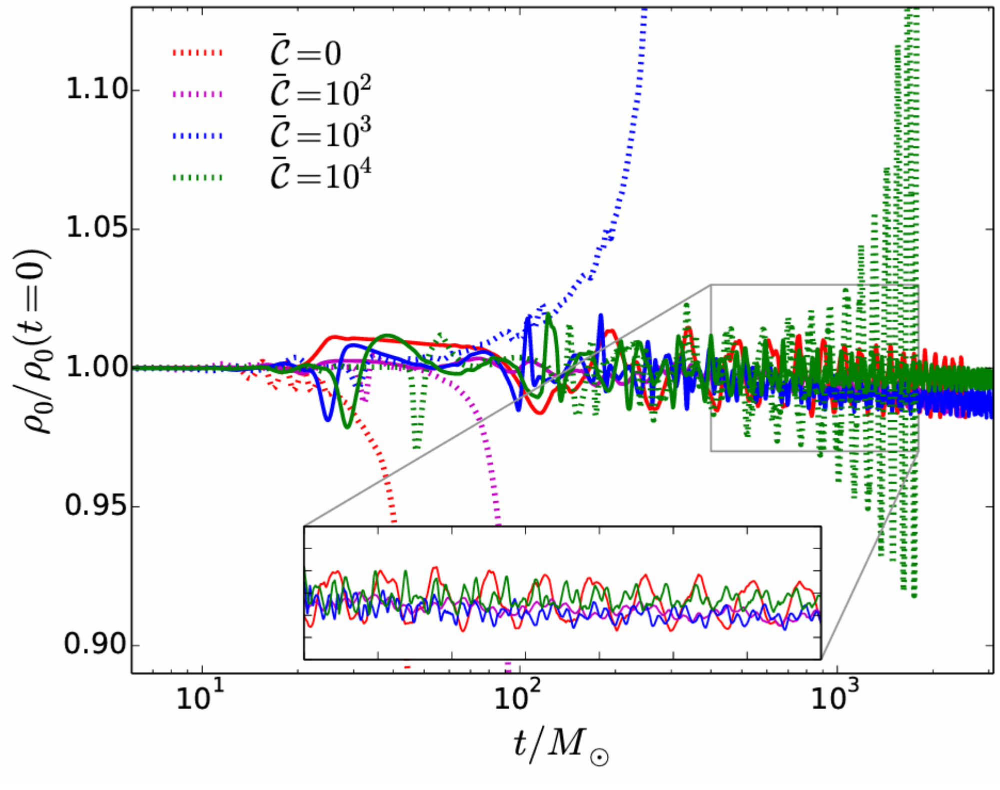
5. Dominant energy condition can break close to the radius

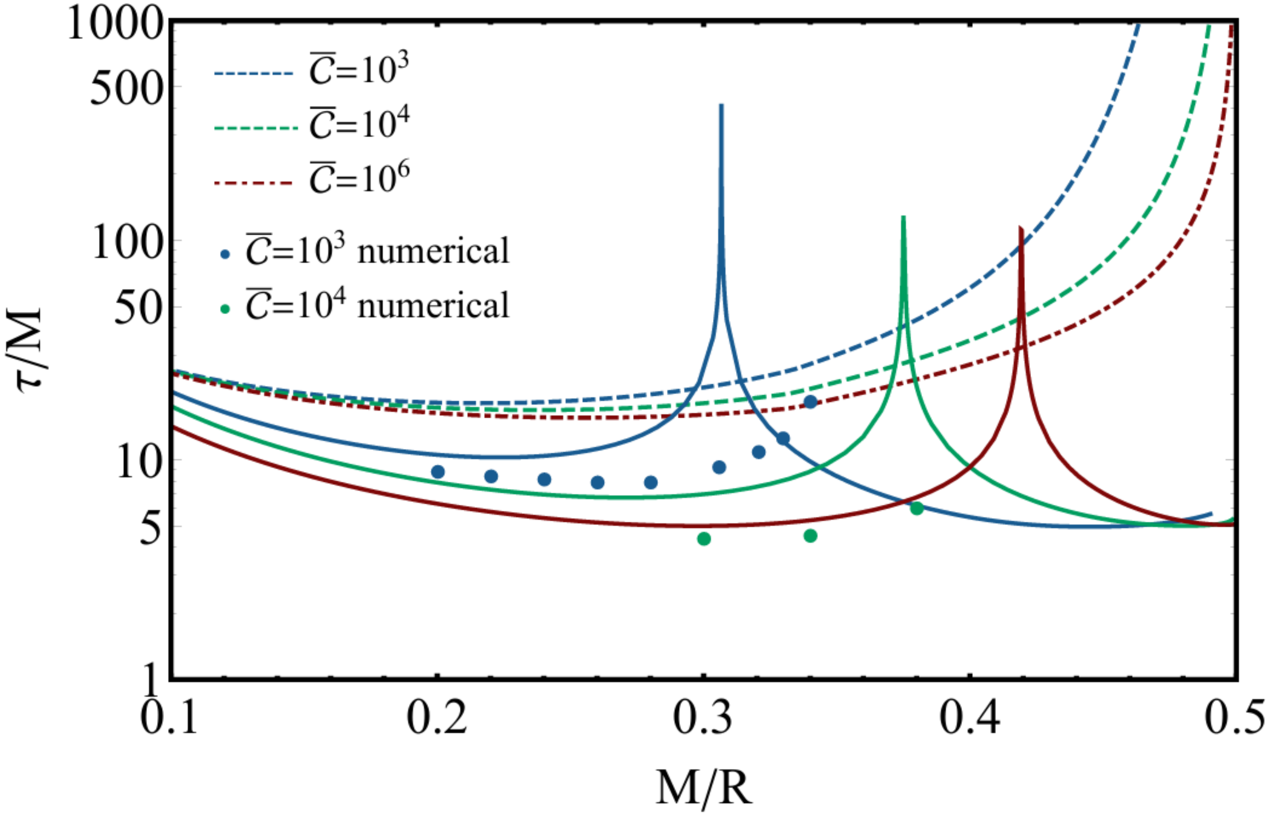


Anisotropic Stars

Covariant Formalism allows to do NR 1+1 evolutions.

Studies of non-linear stability of the star.





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Compact elastic objects in general relativity

Artur Alho,[1] José Natário,[1] Paolo Pani,[2] and Guilherme Raposo[3,4

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[2] *Dipartimento di Fisica, Sapienza Università di Roma & INFN Roma1, Piazzale Aldo Moro*
[3] *CENTRA, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-0*
[4] *Centre for Research and Development in Mathematics and Applications (CIDMA),*
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Self-gravitating anisotropic fluids. I: Context and overview

Tom Cadogan and Eric Poisson
Department of Physics, University of Guelph, Guelph, Ontario, N1G 2W1, Canada
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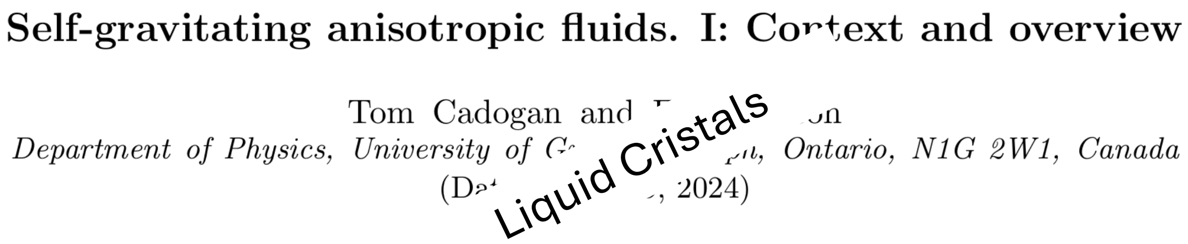
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Relativistic Elasticity

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Liquid Cristals

Same idea: Start from a Lagrangian formalism!

Relativistic Elasticity

A classical **rigid body**:

*Object for which the **distances between points** are constant at **any given instance in time** remains constant.*

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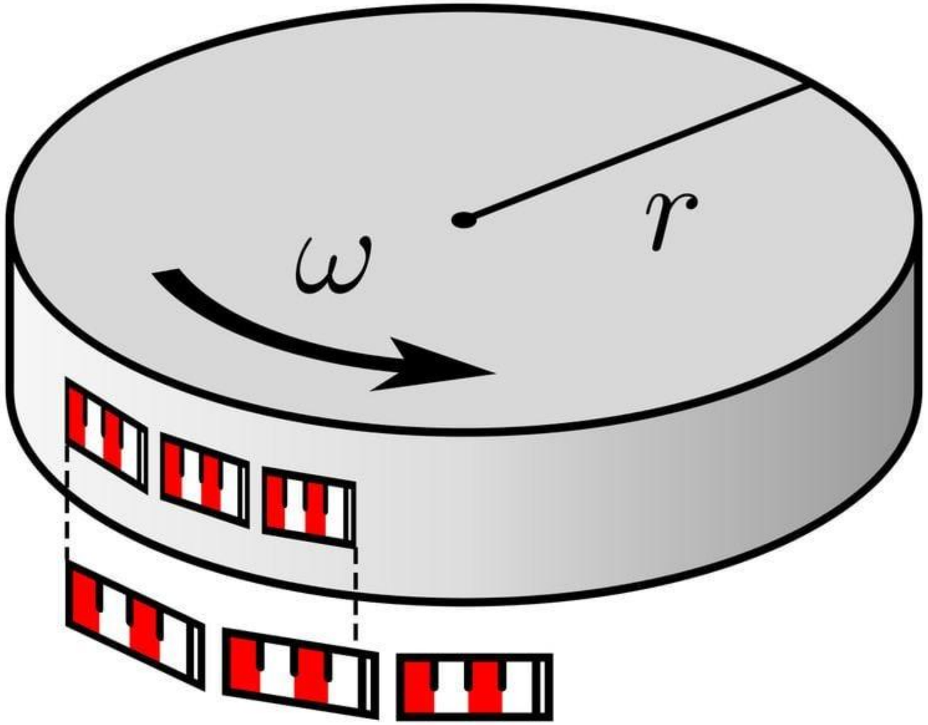
Therefore: There are **no rigid bodies** in relativity!

Physically it **takes some time** for one end of a finite-size body to **receive information** about forces acting on the other end.

Ehrenfest's paradox:



No undeformable bodies in relativity!

Relativistic Elasticity

A bit of theory: (Mostly people in Mathematical Relativity community)

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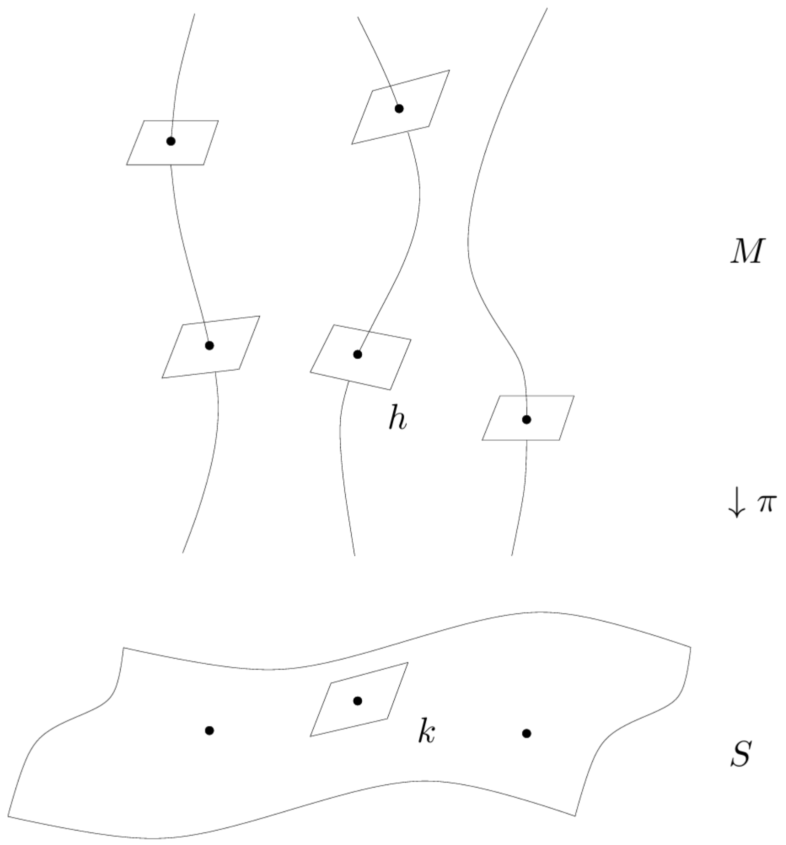
3 key ingredients:

1. Physical spacetime $(\mathcal{M}, g)$
   - Where your deformed object lives.
2. Reference spacetime $(\mathcal{B}, \gamma)$
   - 3-Riemannian manifold - "undeformed body".
3. Projection map: $\Pi : \mathcal{M} \to \mathcal{B}$
   - The level sets of the projection map are the worldlines of the medium particles.



The projection map:

We can make it more concrete by assigning some local coordinates.

$$\Pi : X^I(x^\mu)$$

Another way of thinking: The mapping defines a set of 3 scalar fields that depend on the spacetime coordinates.

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$$H^{IJ}$$

This gives you a definition of **strain**!

$$E^{IJ} = \frac{1}{2}(H^{IJ} - \gamma^{IJ})$$

The reference state:

Let's think about 2D

Preferred **undeformed** state

In a 2 +1 Minkowsky spacetime

Deformed object

The objected has stretches and deforms due to its natural preferred state.

Relativistic Elasticity

Our set of equations (from GR):

We choose a Lagrangian density of the type.

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$$\mathcal{L} = T_{\bar{0}\bar{0}} = \rho$$

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The choice of $\rho = \rho(X^I, H^{IJ})$ corresponds to the choice of an **elastic law!**

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Once we have the Lagrangian we can obtain the **stress-energy tensor**.

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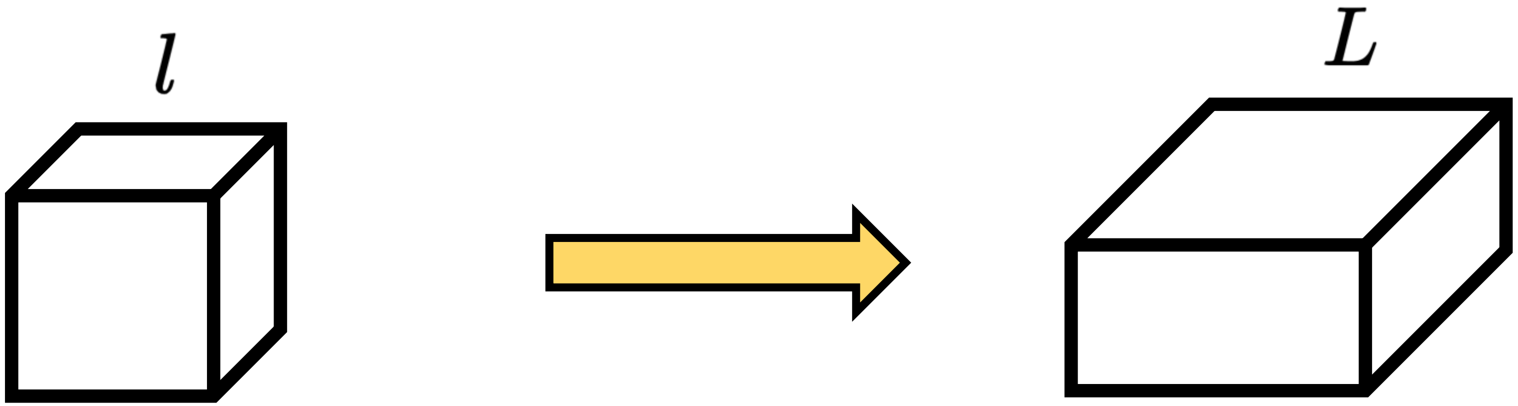
1. **Homogeneous materials:** The Lagrangian (EoS) does not depend on the positions.
2. **Isotropic materials:** The Lagrangian (EoS) depends only on the deformation $\mathcal{H}^I_J$ through its eigenvalues, specifically the principal invariants.

Physical meaning:

Eigenvalues of $\mathcal{H}^I_{\ J}$ tell you how much the principal directions of your material stretch when they are deformed.



$$h \sim L/l$$

Relativistic Elasticity

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Equivalently: **Linear densities** along the principal directions.

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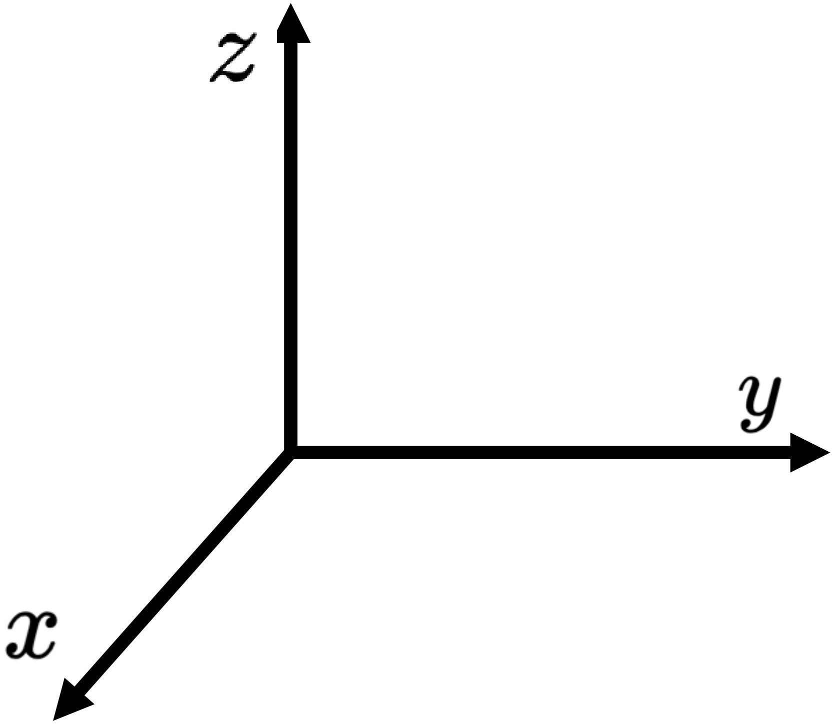
The relativistic elasticity theory tells you how to compute the pressures from the EOS.
No need for additional ad-hoc EoS.

Relativistic Elasticity

The formalism algo gives you expression for the speeds of sound!



Two types of waves:

- **Longitudinal waves**

$$c_{\mathrm{L}i}^2 = \frac{n_i \dfrac{\partial p_i}{\partial n_i}}{\rho + p_i} = \frac{n_i^2 \dfrac{\partial^2 \rho}{\partial n_i^2}}{\rho + p_i}$$

- **Transverse waves:**

$$c_{\mathrm{T}ij}^2 = \begin{cases} \dfrac{n_j^2}{\rho + p_j} \dfrac{(p_j - p_i)}{n_j^2 - n_i^2}, & \text{if } n_i \neq n_j, \\[2ex] \dfrac{\frac{1}{2} n_j}{\rho + p_j} \left(\dfrac{\partial p_i}{\partial n_i} - \dfrac{\partial p_j}{\partial n_i} \right), & \text{if } n_i = n_j. \end{cases}$$

For anisotropic stars there was **no formalism** to compute **these sound speeds**! Affects **causality** studies!

Einstein's Elastic Equations

Turns out that the stress-energy tensor is exactly the same as the anisotropic fluid.

Same system of **anisotropic TOV equations**.

$$m' = 4\pi r^2 \rho \qquad \phi' = \frac{m + 4\pi r^3 P_r}{r(r-2m)}$$

Same as perfect-fluid

$$P_r' = -(\rho + P_r)\phi' - \frac{2}{r}(P_r - P_t)$$

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Introduce our EoS: $\rho = \rho(\delta, \eta)$ → More convenient combination of "principal linear densities"

Einstein's Elastic Equations

Turns out that the stress-energy tensor is exactly the same as the anisotropic fluid.

Same system of **anisotropic TOV equations**.

$$m' = 4\pi r^2 \rho \qquad \phi' = \frac{m + 4\pi r^3 P_r}{r(r-2m)}$$

Same as perfect-fluid

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$\delta = n_1 n_2^2$ "Number density of particles"

$\eta(r) = n_2^3 = \frac{3}{r^3}\int_0^r \frac{\delta(u)u^2 du}{(1-2m(u)/u)^{1/2}}$ "Average number density of particles"

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Everything depends on δ ! The system is now closed!

Quadratic EoS

With the formalism set, the question reduces to prescribe an EoS for elastic matter.

Start with the simplest case: A **polytropic** with a quadratic **elastic** correction!

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Our elastic EOS: Fluid polytrope EoS + **quadratic elastic correction**.

$$\rho = \varrho + n\mathcal{K}\varrho^{1+1/n} + \mathcal{E}\mathcal{K}^n(\varsigma - \varrho)^2$$

$\varrho \sim \delta$, $\quad (\varsigma - \varrho) \sim \eta$

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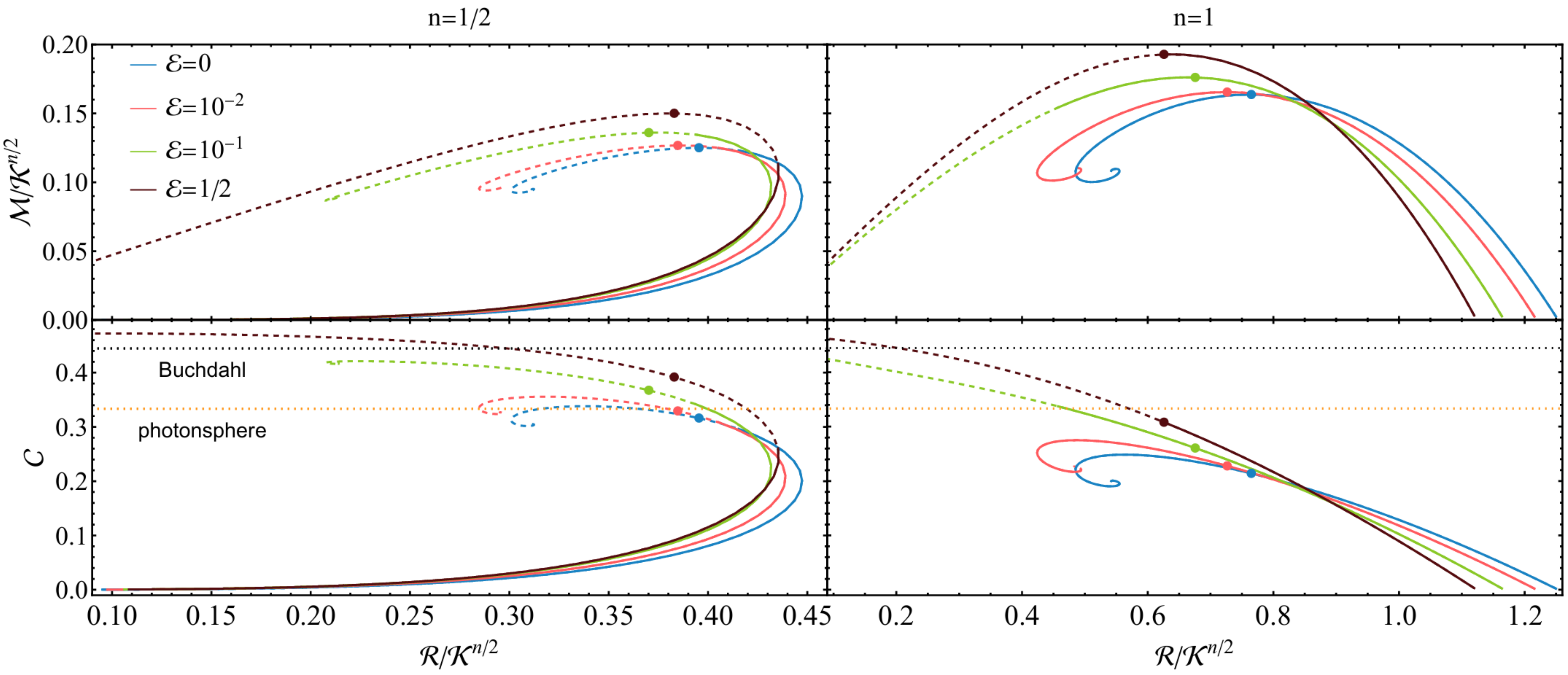
$\varrho \sim \delta$, $(\varsigma - \varrho) \sim \eta$

Note:
These new variables make the system **invariant** with respect to the **reference state.**

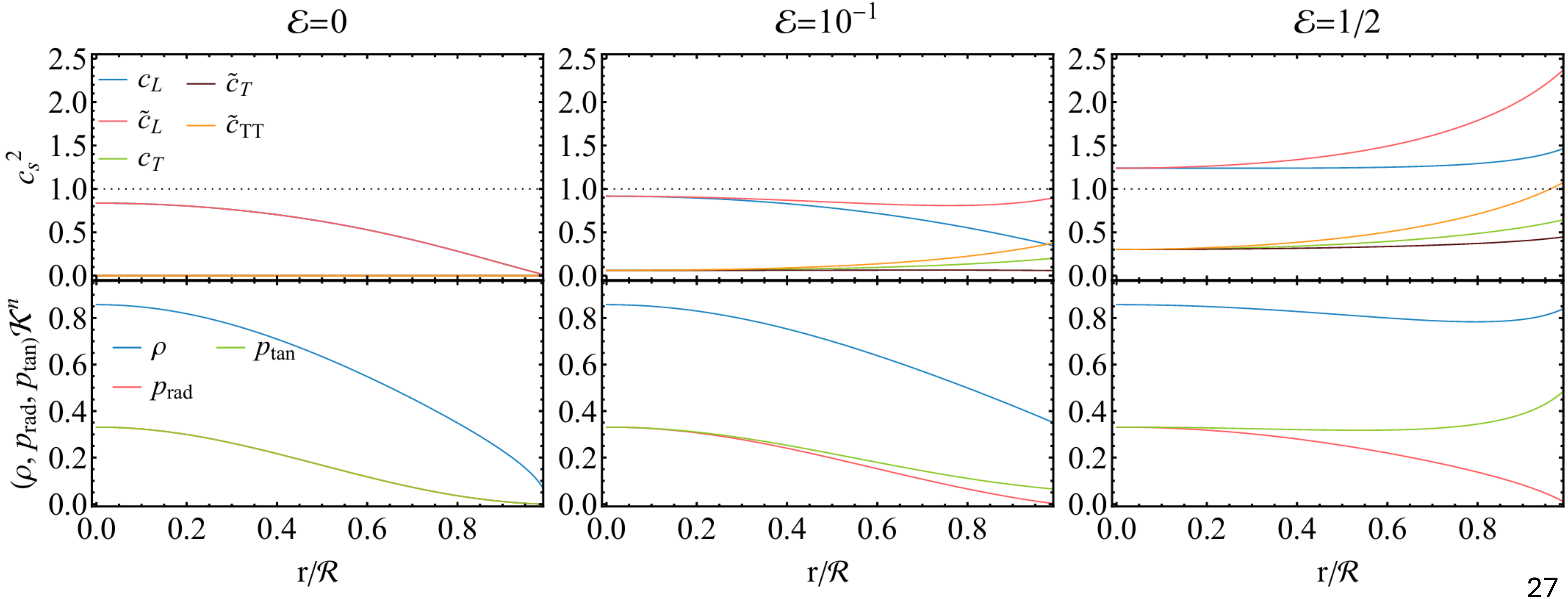
Quadratic EoS - Results



Quadratic EoS - Results

In spherical symmetry there are only **5 independent sound speeds**!

Actually, we need to specify the form of one the sound speeds using a "natural choice".



Affine constant sound speed EoS

From last lecture:

Constant sound speed EOS (affine):

$$\rho = \frac{\gamma - 1}{\gamma}\rho_0 + \frac{K}{\gamma - 1}\varrho^{\gamma}$$

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However: We don't have freedom to set all speeds constant. We **set constant longitudinal wave speed!**

$$c_{\mathrm{L}i}^2 = \frac{n_i \frac{\partial p_i}{\partial n_i}}{\rho + p_i} = \frac{n_i^2 \frac{\partial^2 \rho}{\partial n_i^2}}{\rho + p_i}$$

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Our elastic affine constant sound speed EoS:

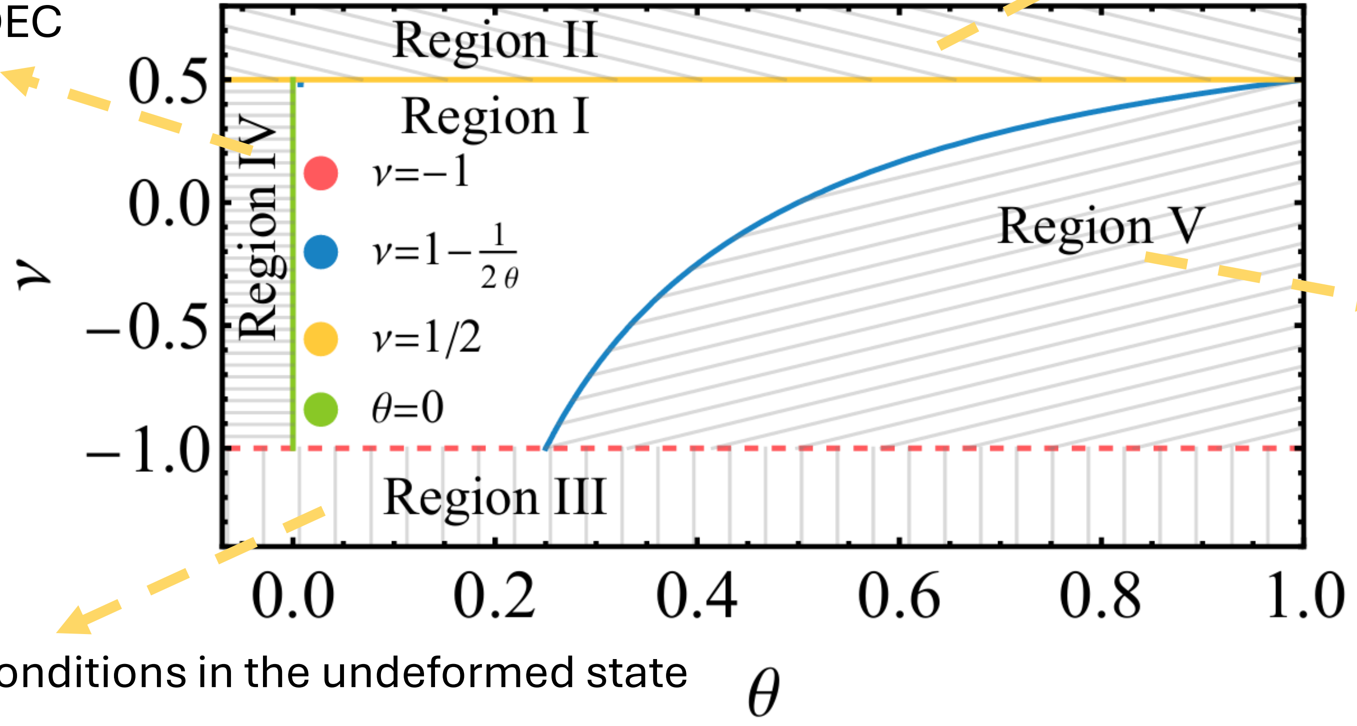
$$\frac{\gamma\widehat{\rho}(\delta,\eta)}{L} = \frac{\gamma}{\gamma - 1} + \left(\frac{1-(2-\gamma)\nu}{\gamma(\gamma-1)(1-\nu)} - \theta\right)(\delta^{\gamma} - 1) + 3\left(\frac{1}{\gamma}\left(\frac{1-2\nu}{1-\nu}\right) + \theta\eta^{\frac{\gamma}{3}}\right)\left(\eta^{\frac{\gamma}{3}} - 1\right)$$
$$+ \eta^{\frac{\gamma}{3}}\left(\frac{1}{\gamma}\left(\frac{1-2\nu}{1-\nu}\right) + \theta\left(2\eta^{\frac{\gamma}{3}} - 1\right)\right)\left(\left(\frac{\delta}{\eta}\right)^{\gamma} - 1\right). \qquad (339)$$

Affine constant sound speed EoS

Realistic physical conditions on the matter restrict the parameter space.

Imaginary sound speeds (transverse waves) in undeformed state

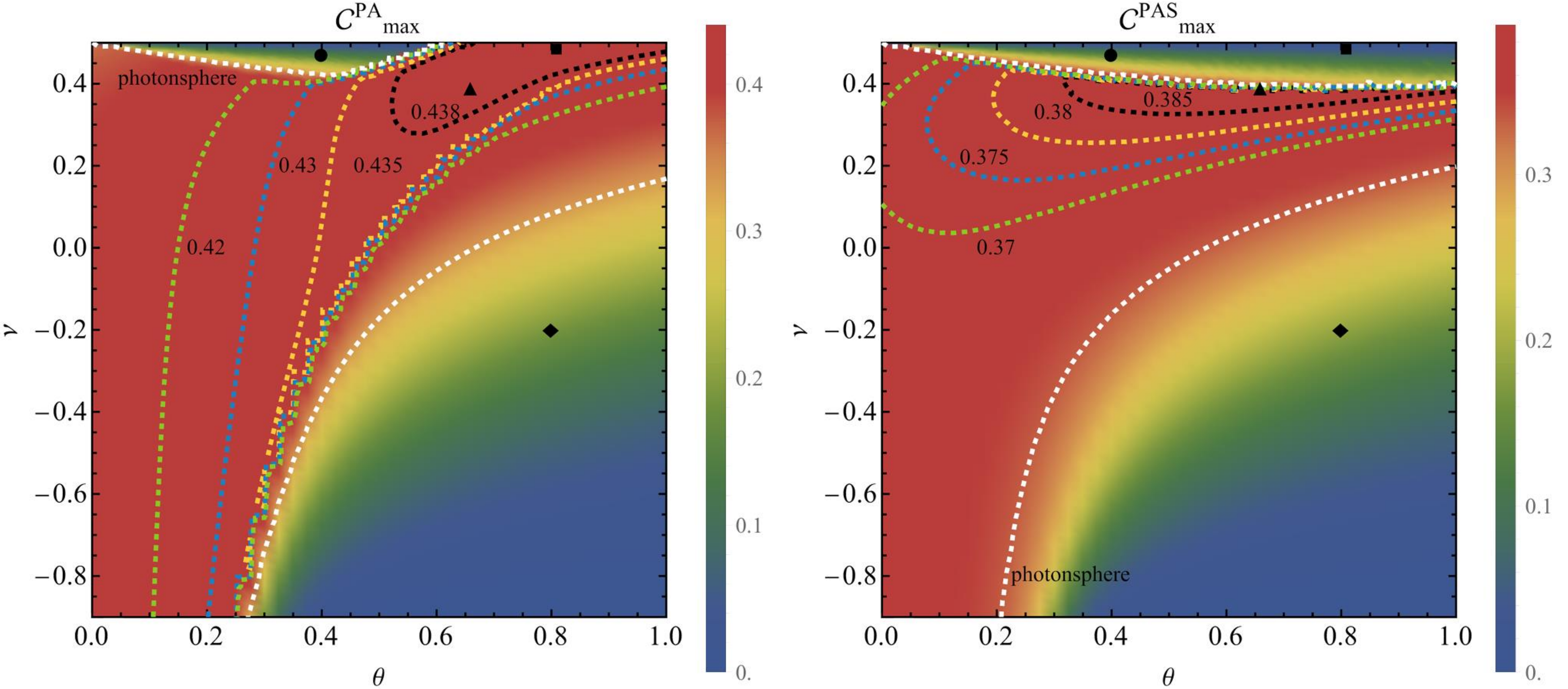
Imaginary sound speeds (transverse waves), DEC



Imaginary sound speeds (transverse waves), negative densities and pressures for large densities.

Breaks elastic conditions in the undeformed state

Affine constant sound speed EoS



$\mathcal{C}^{\mathrm{PA}}_{\mathrm{max}} \lesssim 0.443$

$\mathcal{C}^{\mathrm{PAS}}_{\mathrm{max}} \lesssim 0.384$

Maximum Compactness of Stars

With **constant sound speed EoS** we can look for bounds on **viable** stars!

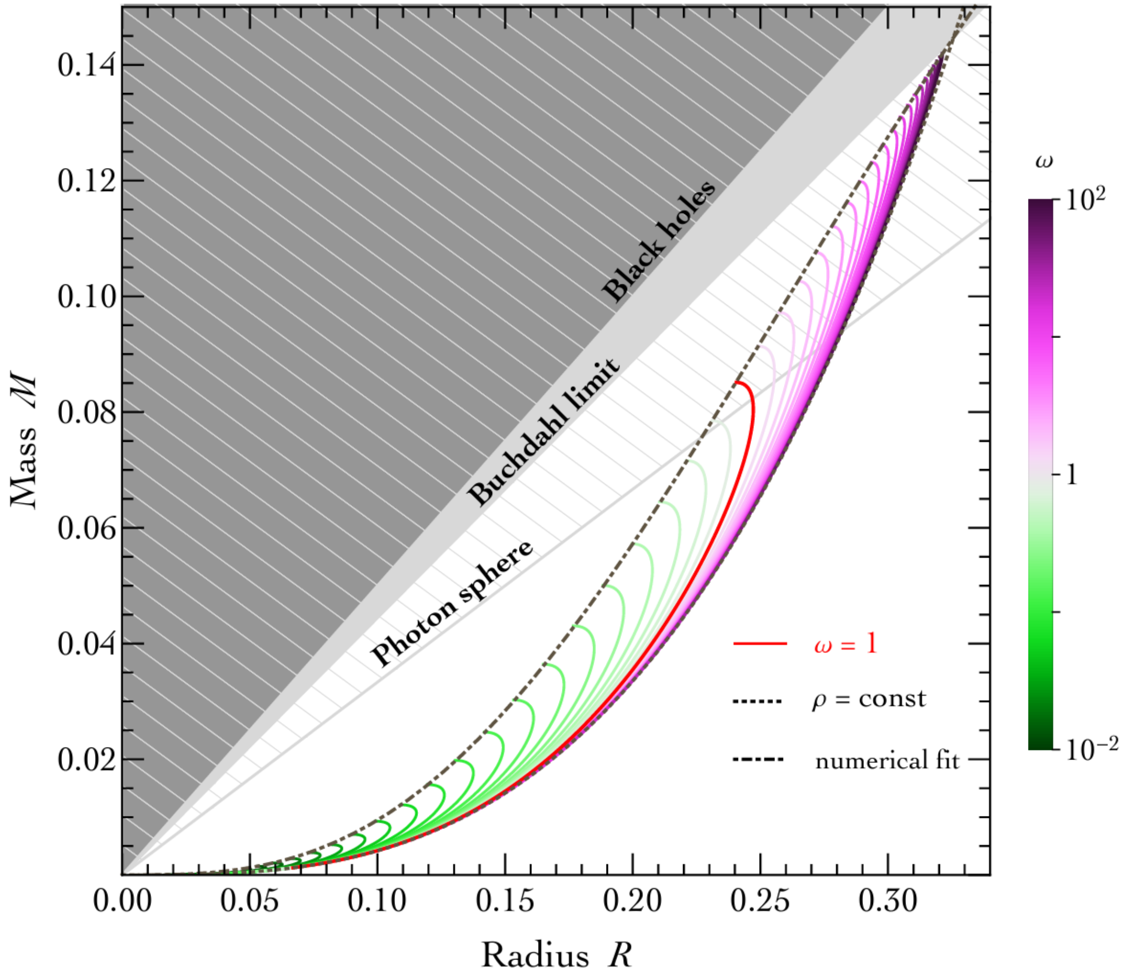
Black Hole: $C = 0.5$

Buchdahl Bound: $C = 4/9$

Causal Buchdahl Bound: $C = 0.364$

Causal Buchdahl bound + Radial Stability:

$$C = 0.354$$



[Urbano & Vermaee, 1810.07137]

Maximum Compactness of Stars

With **constant sound speed EoS** we can look for bounds on **viable** stars!

Black Hole: $C = 0.5$

Buchdahl Bound: $C = 4/9$ } Superluminal wave propagation

Causal Buchdahl Bound (fluid): $C = 0.364$

Maximum Compactness for Physically Admissible stars (elastic): $\mathcal{C}_{\max}^{\mathrm{PA}} \lesssim 0.443$

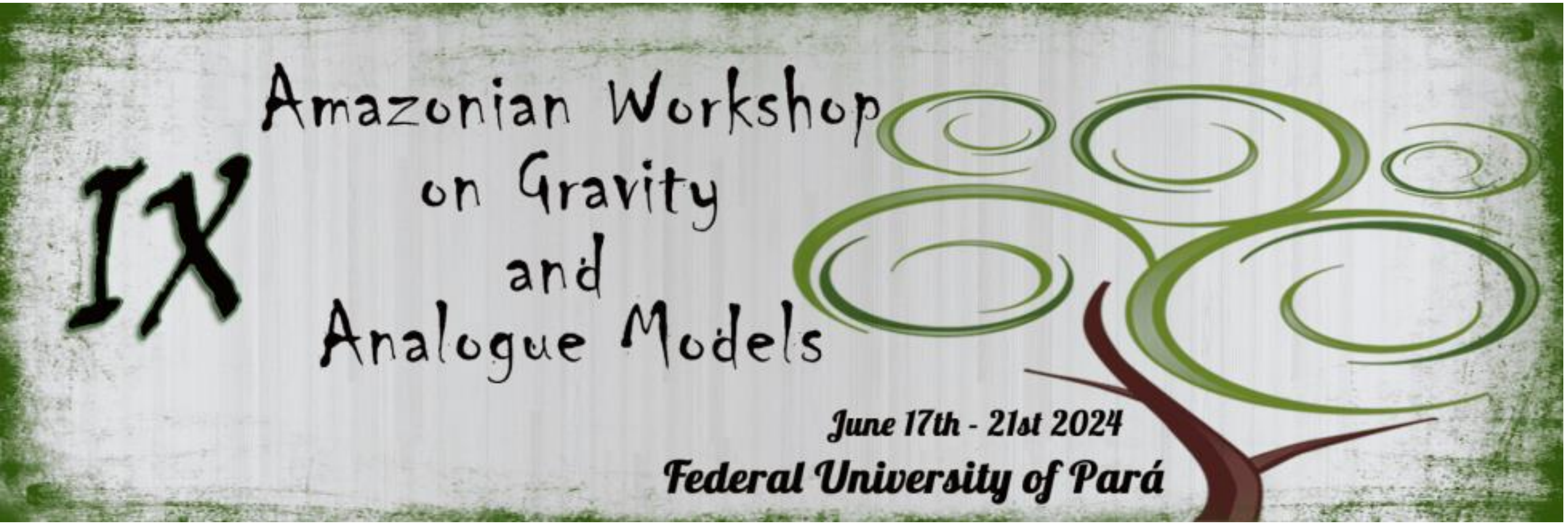
} Causal & Physically Admissible

Causal (stable) Buchdahl bound (fluid): $C = 0.354$

Bound (stable) for Phys. Admissible stars (elastic): $\mathcal{C}_{\max}^{\mathrm{PAS}} \lesssim 0.384$

} Causal & Physically Admissible & **Radially Stable**

Compact Objects and How to Model Them
Part II

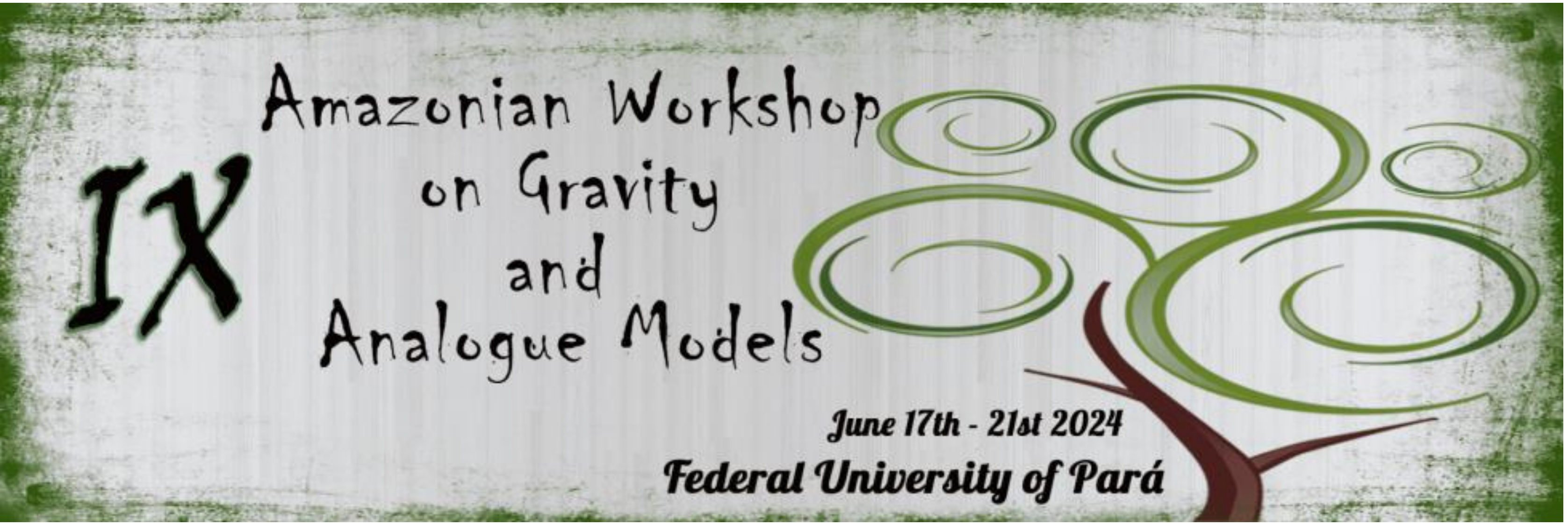


Guilherme Raposo

Universidade de Aveiro (Gr@v)

18/06/2024

Compact Objects and How to Model Them
Part III



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Last lectures in a nutshell

Part I:

Compact objects in our Universe: Black Holes, Neutron Stars and White Dwarfs.

Fluid models for compact objects: TOV equations.

Equation of state:

- Constant density;
- Polytropes;
- Constant sound speed;

Buchdahl limit and Causal Buchdahl limit;

Last lectures in a nutshell

Part II:

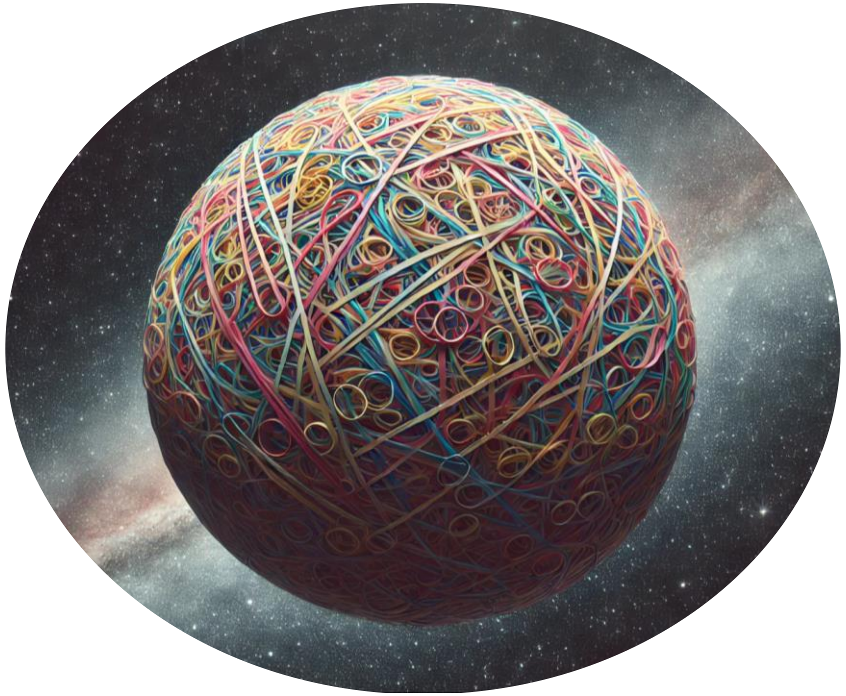
Compact objects and Exotic Compact Objects;

Why we care about Exotic Compact Objects;

How to construct different models of ECOs:

"Artificial" models (wormholes, gravastars, etc...)

Anisotropic stars & Elastic Stars



Last lectures in a nutshell

Part II:

Anisotropic Stars:

- Constructed by solving a system of **anisotropic TOV equations.**
- Problems: Additional **ad-hoc EoS**; Spherically symmetry;

Elastic Stars:

- Similar idea but start from **Lagrangian** approach.
- Leads to the same system of **anisotropic TOV equations.**
- Relasticity tells you how to obtain the pressures from the EoS. **No** need for **additional ad-hoc EOS.**
- Does not require necessarily spherically symmetry and is covariant naturally.

Key features:

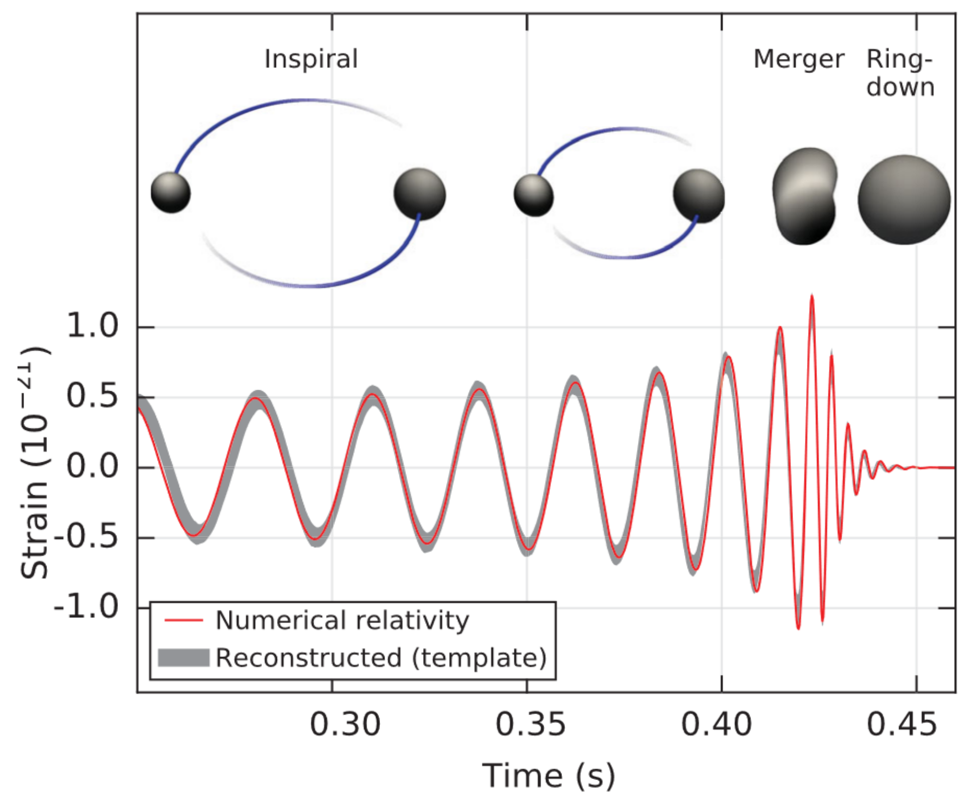
Allows to construct ultracompact **physically viable** objects.

How to do it in practice

Inspiral Phase:

- Multipole Moments;
- Tidal heating;
- Tidal deformations;

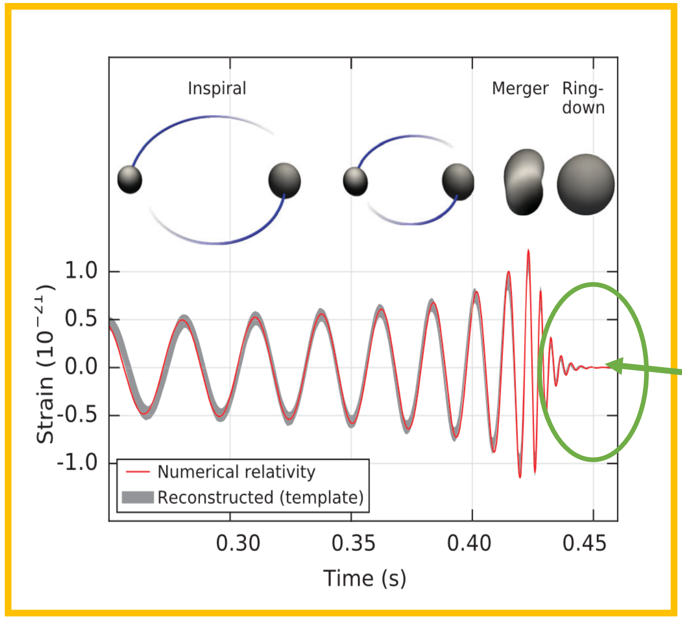
Phenomenology

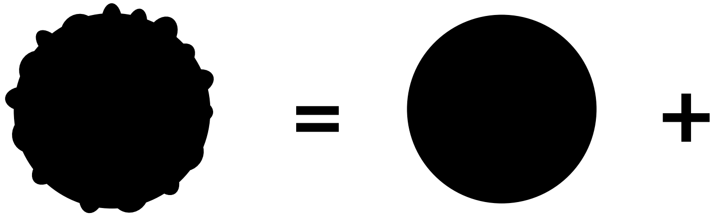


Post-Merger:

- Quasinormal Modes;
- Gravitational Echoes;

Post Merger





Post-Merger Spacetime = **Final Stationary Spacetime** + **Time-dependent Perturbation**

Perturbation is governed by:

$$-\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial z^2} - V\psi = 0$$

Where:

$$V_{\text{axial}} = f\left(\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right),$$

$$V_{\text{polar}} = 2f\left(\frac{q^2(q+1)r^3 + 3q^2Mr^2 + 9M^2(qr+M)}{r^3(qr+3M)^2}\right),$$

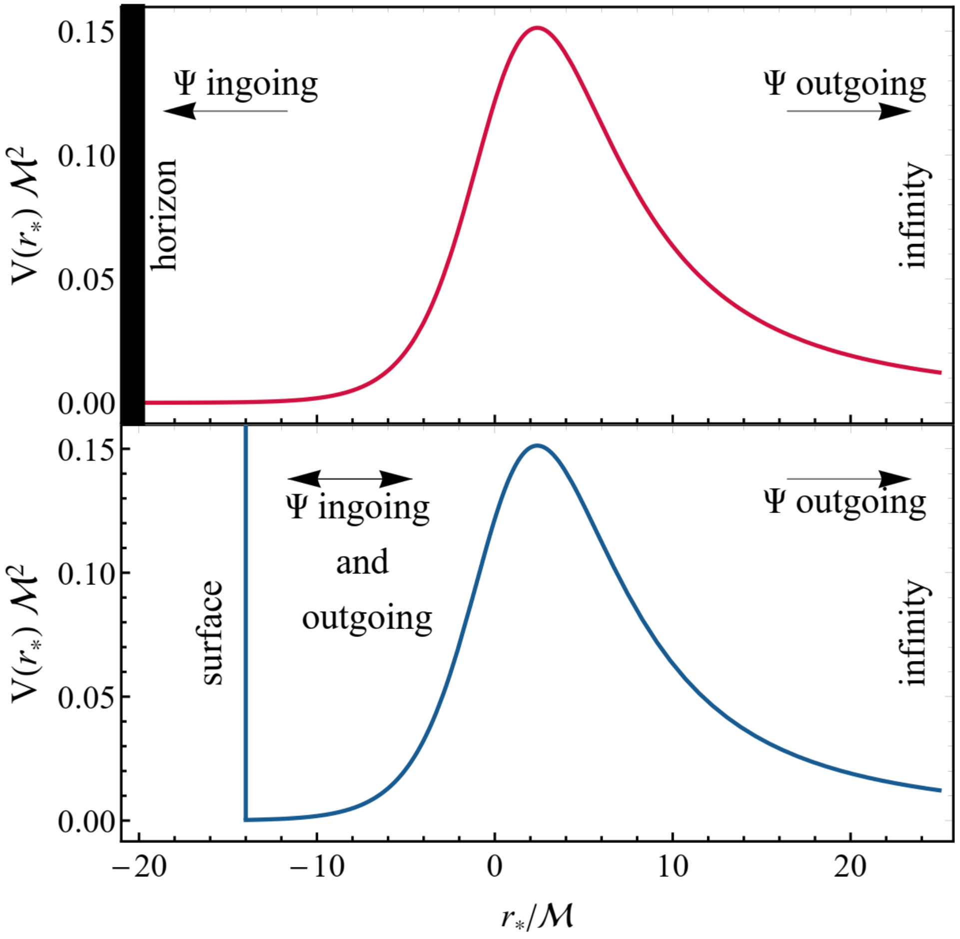
[Regge, Wheeler, 1957 & Zerilli, 1970]

Time independent version:

$$\frac{d^2\psi(r)}{dr_*^2} + \left[\omega^2 - V(r)\right]\psi(r) = 0$$

Post Merger

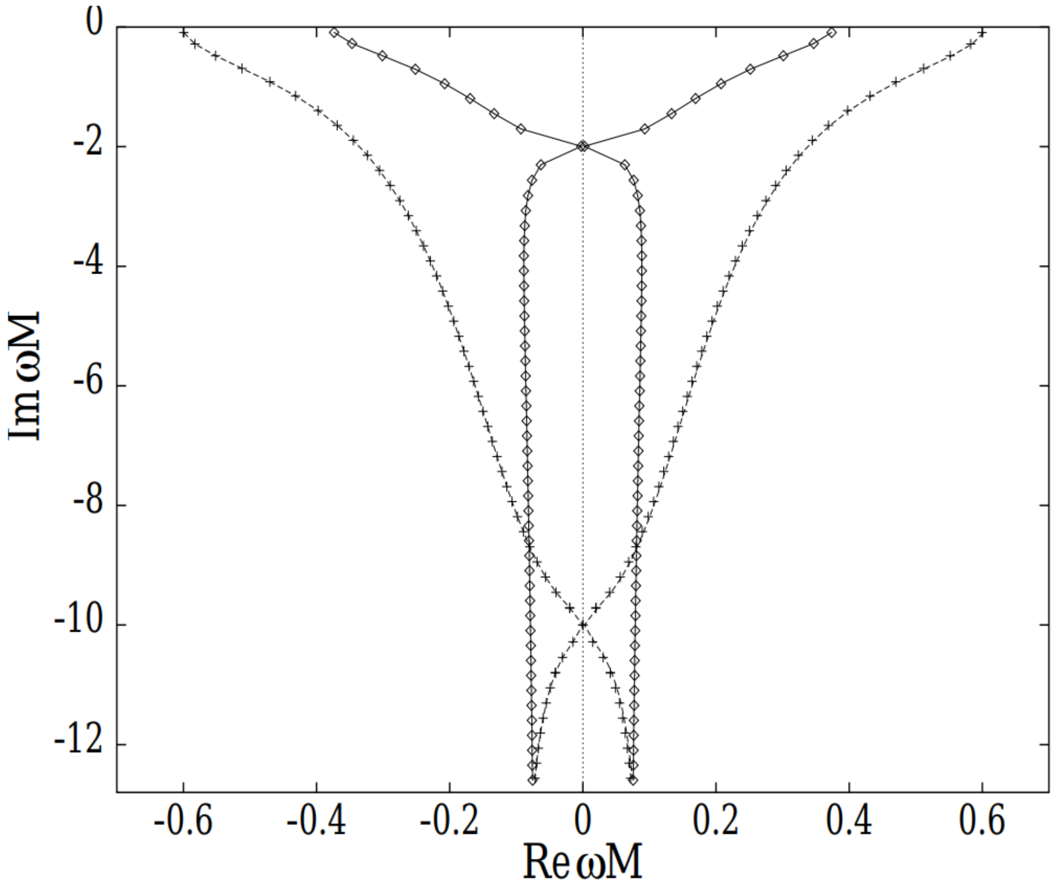
Applying appropriate **boundary conditions** at the **surface/horizon** and at **infinity** leads to an **eigenvalue problem** for the frequency.



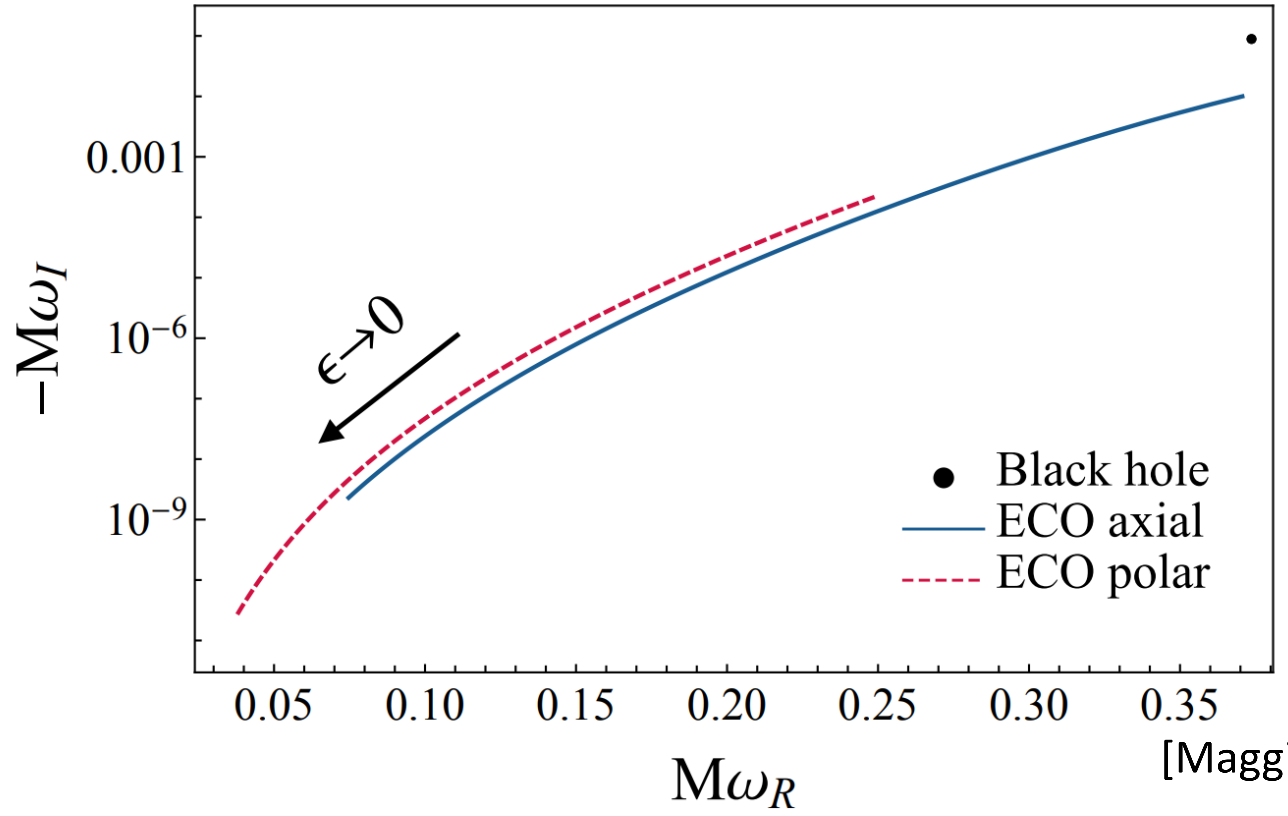
QNMs of a BH

Properties of the BH QNMs:

- Isospectrality.
- Imaginary part becomes increasingly larger with n.
- High overtones have quick damping time.



QNMs of an ECO



[Maggio, Pani, Raposo, 2021]

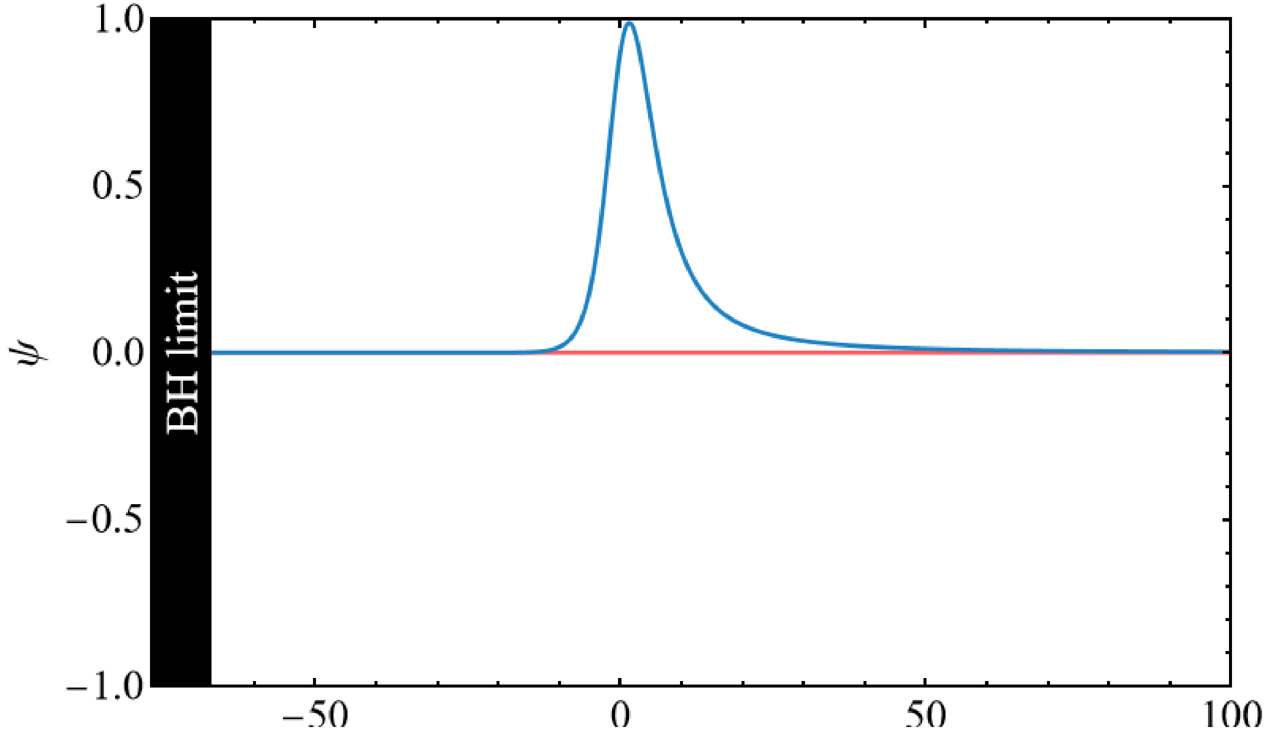
Properties of the ECO QNMs:

- Breaking of isospectrality;

[Chandrasekar, Detweiler, 75]

- In the BH limit the ECO QNMs are low-frequency and long-lived.

Time-Dependent

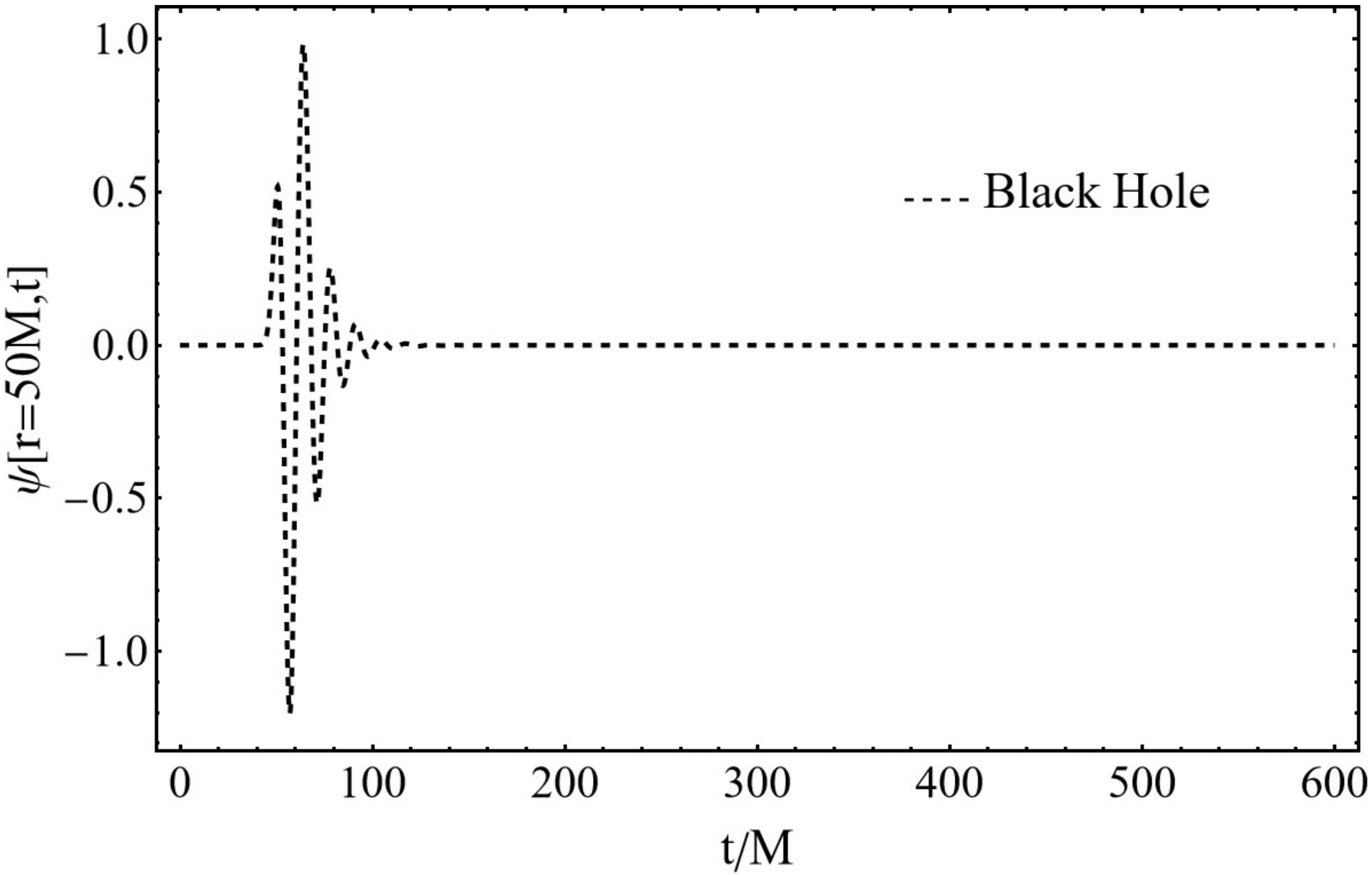


[Maggio, Pani, Raposo, 2021]

BH case:

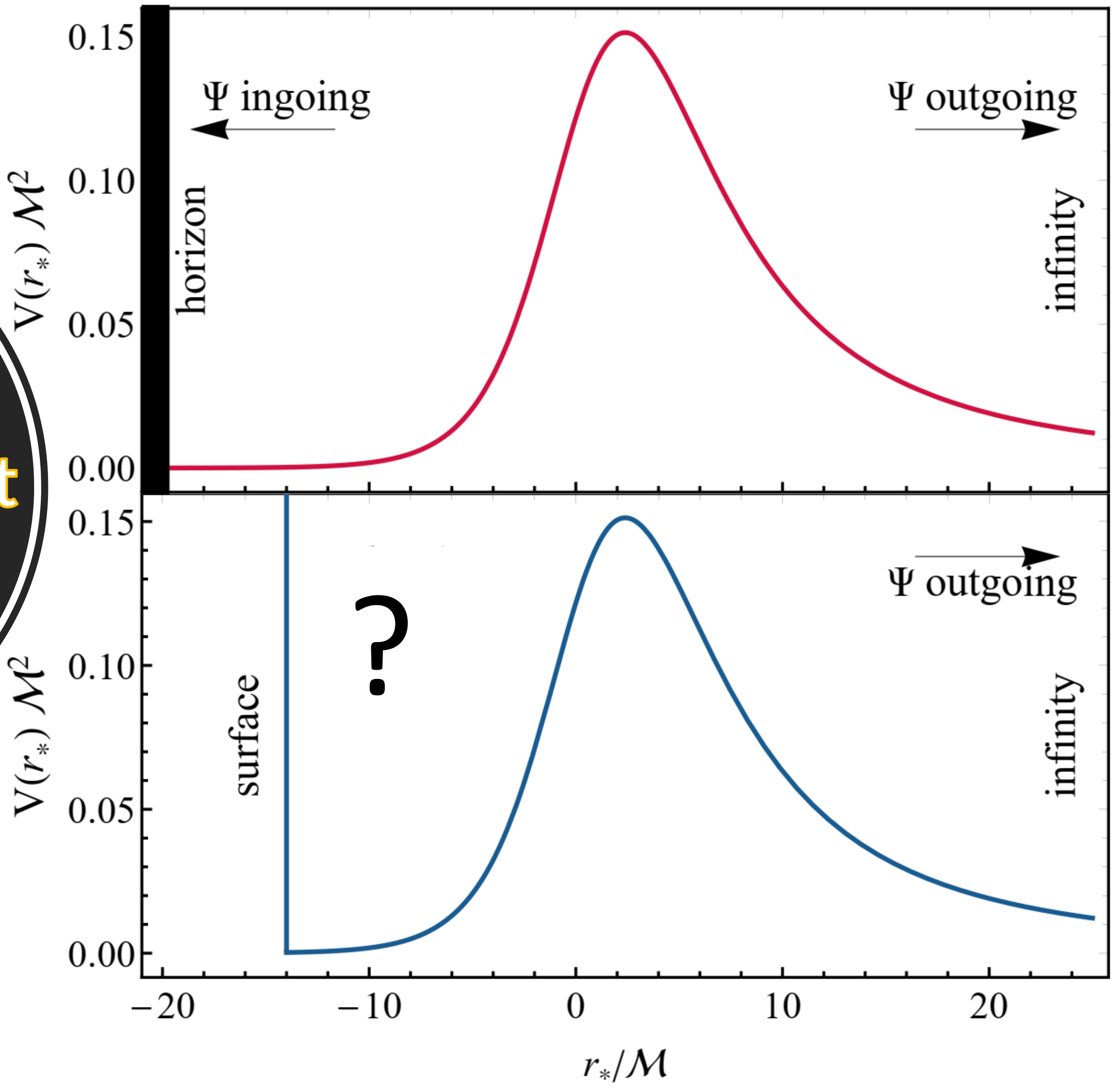
- Perturbation interacts with potential maximum (close to photonsphere).
- Perturbation splits into two contributions.
  - Reflected to infinity.
  - Transmitted towards horizon.

Time-dependent



$$e^{-0.0898t/M} \sin(0.374t/M)$$

Time-dependent

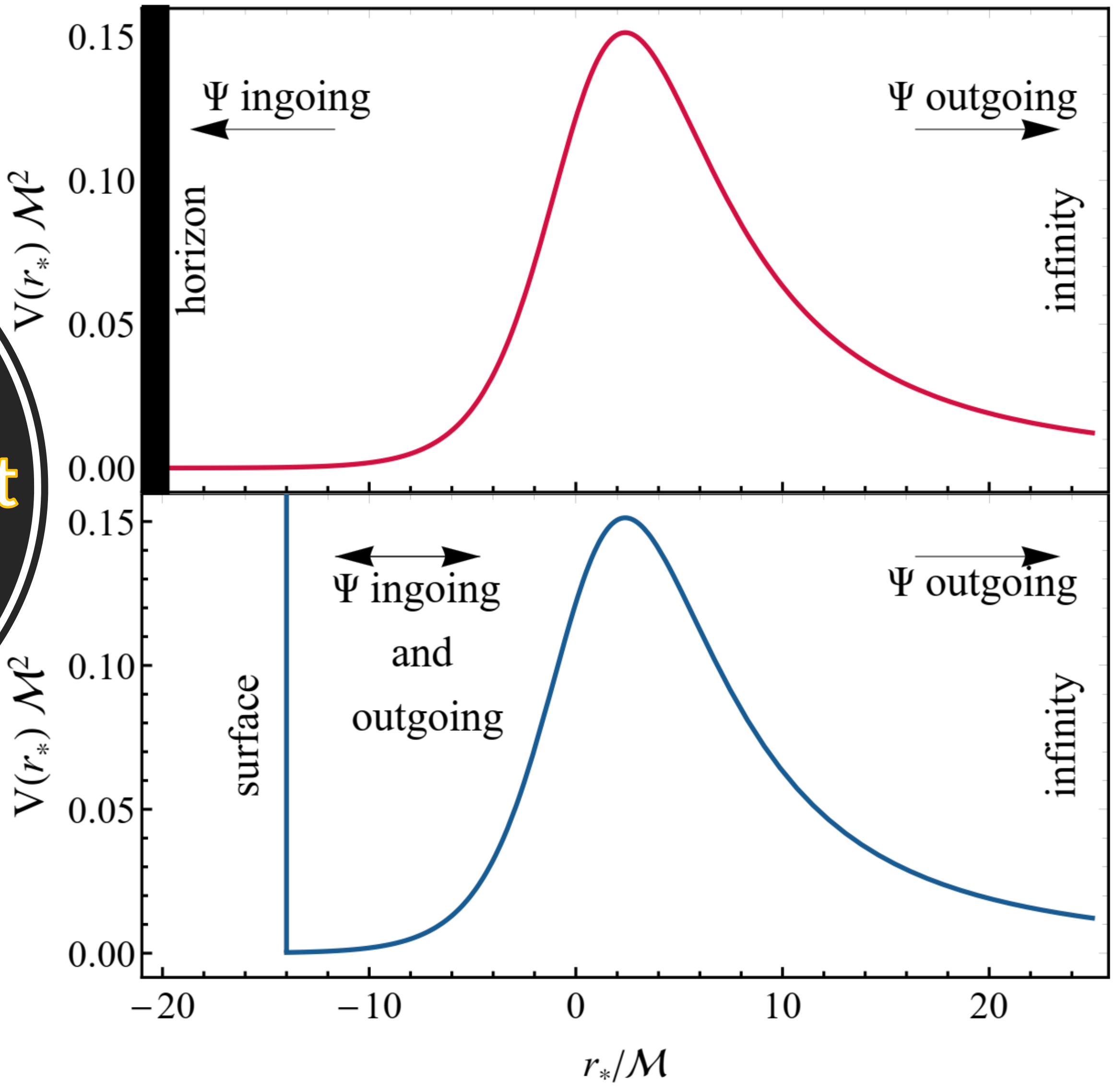


BH case:

- Ingoing wave – absorbed at horizon.

ECO case:

Time-dependent



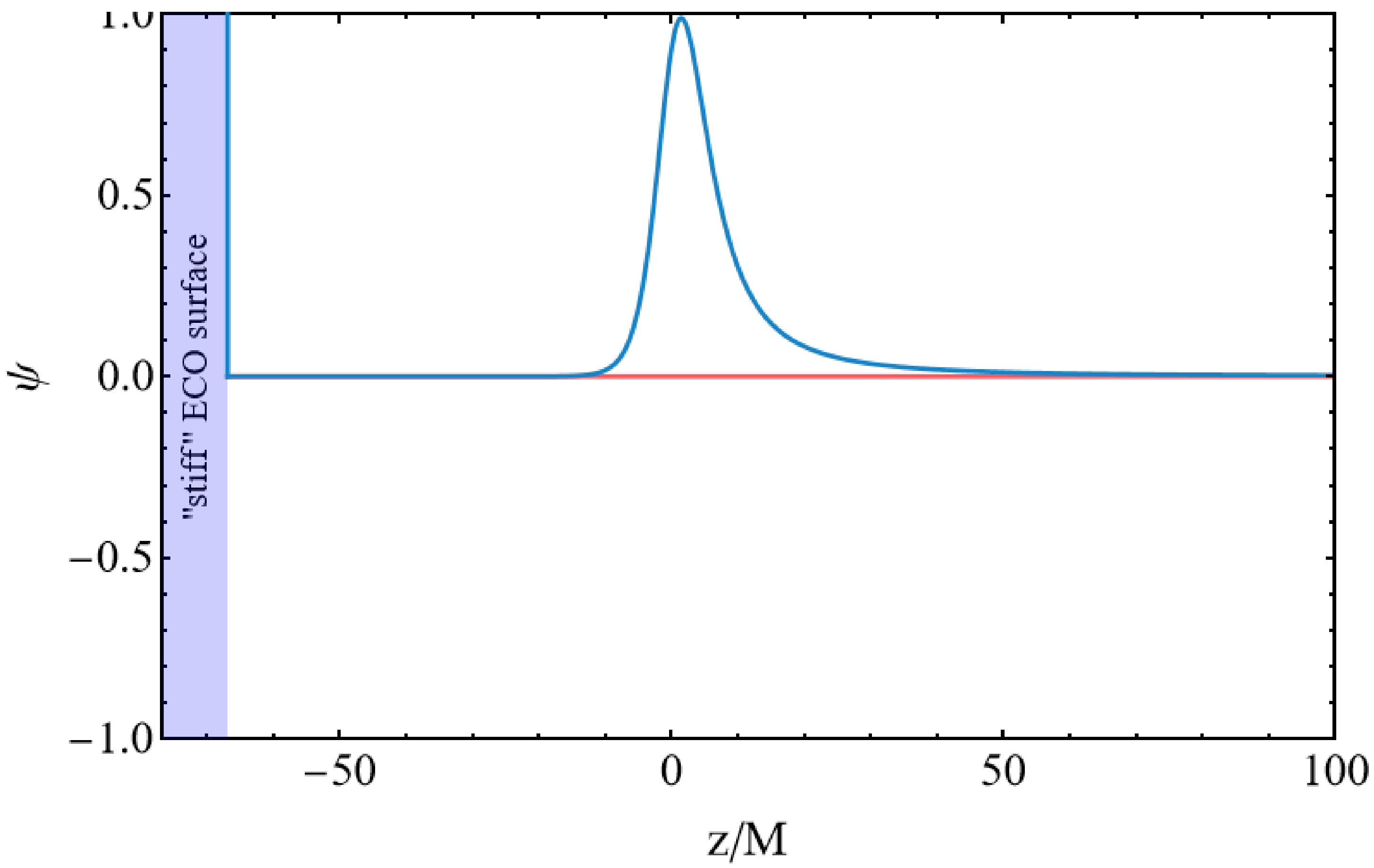
BH case:

- Ingoing wave – absorbed at horizon.

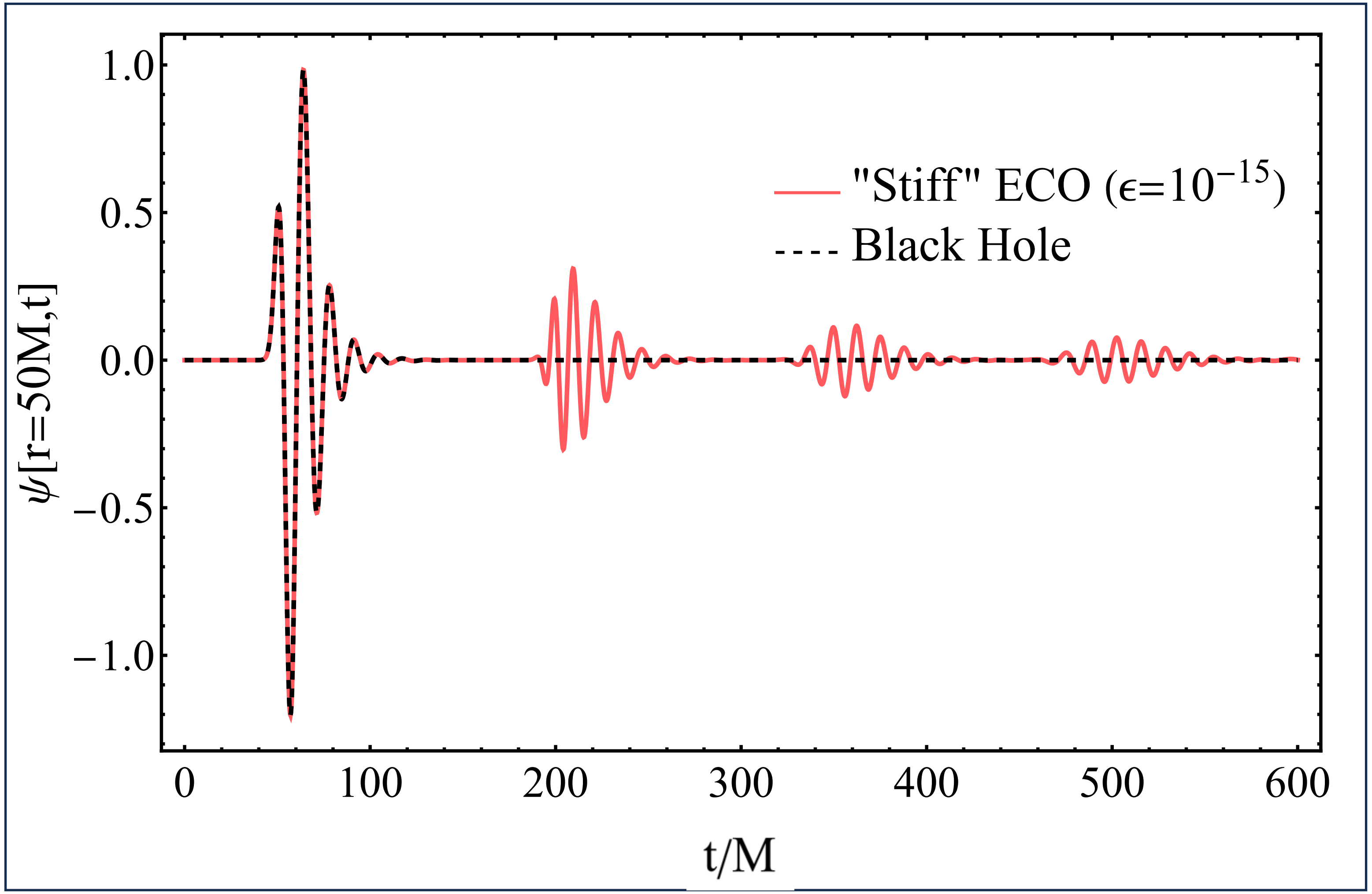
ECO case:

- Mix of ingoing and outgoing wave.
- Waves are reflected between the potential wall at surface and at potential maximum.

Time-dependent



Echoes



Prompt Ringdown:

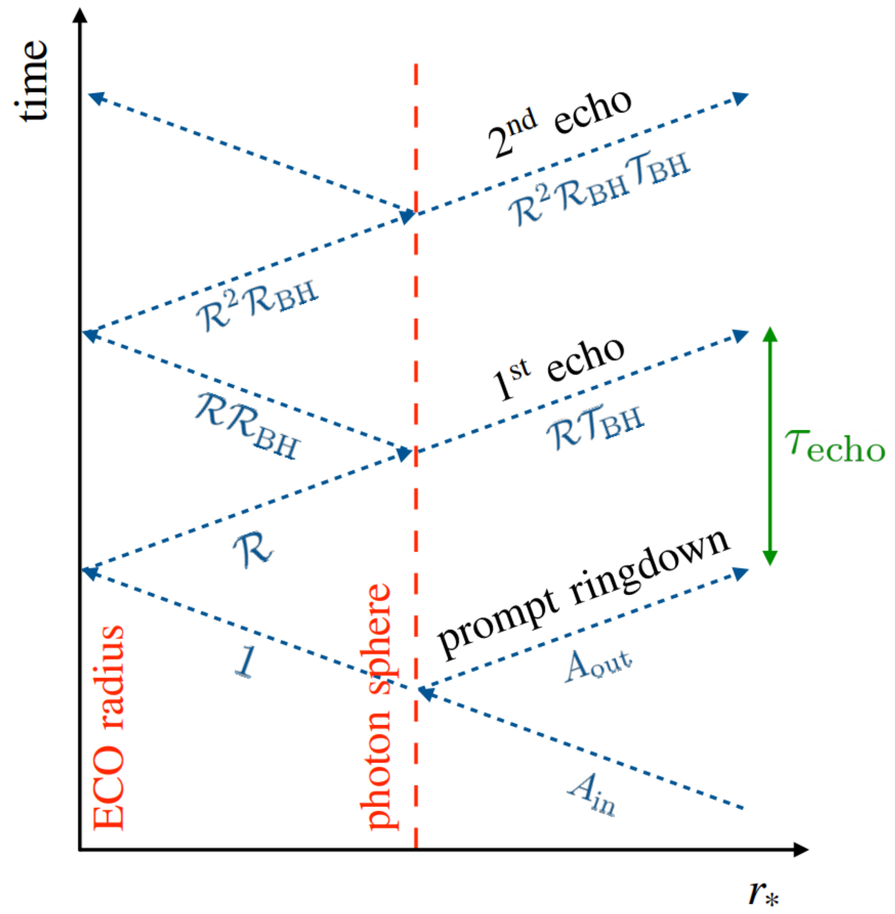
- Same signal, BH and ECO. Why?
- Where do you observe the QNMs in this case?

Echoes

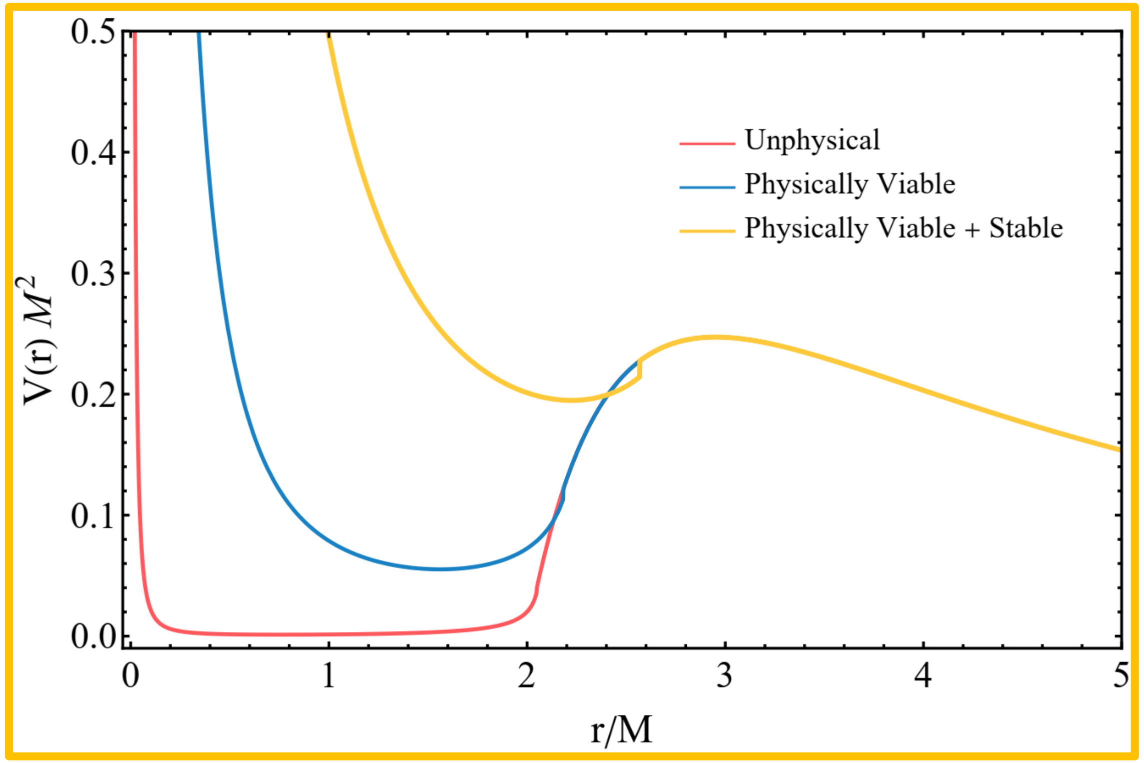
Prompt Ringdown:

- The ringdown has no information on the boundary/surface.

- The information on the surface appears at later times.

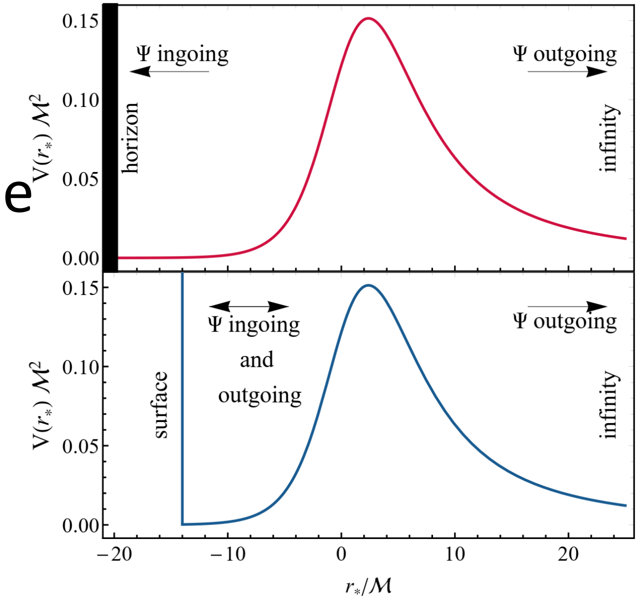


Echoes Ultracompact Stars

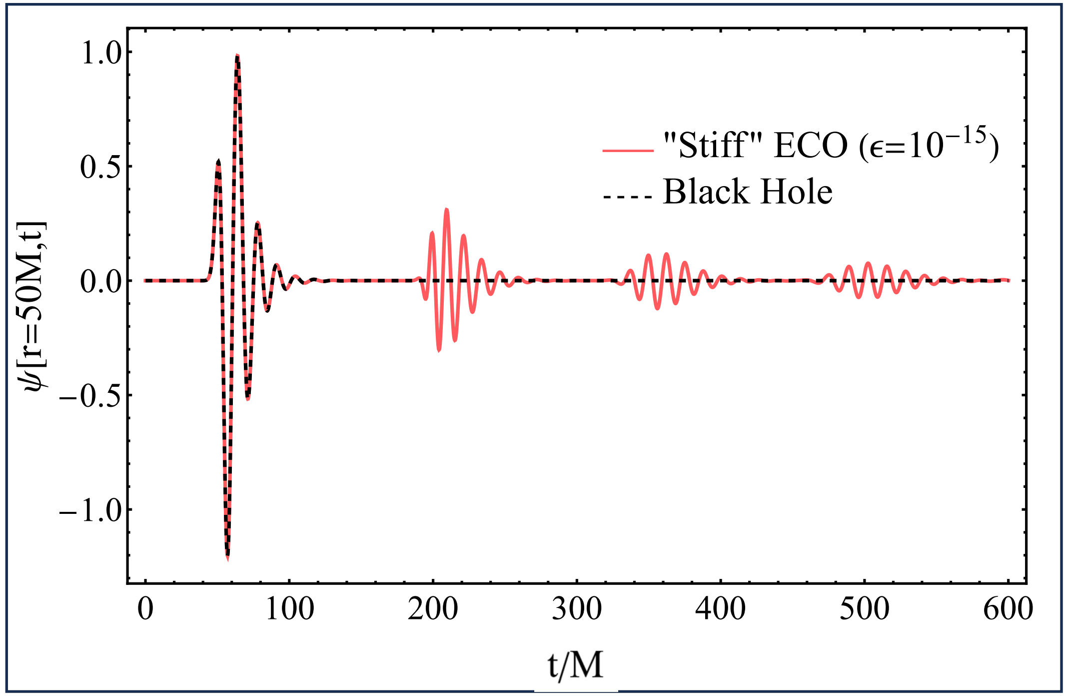


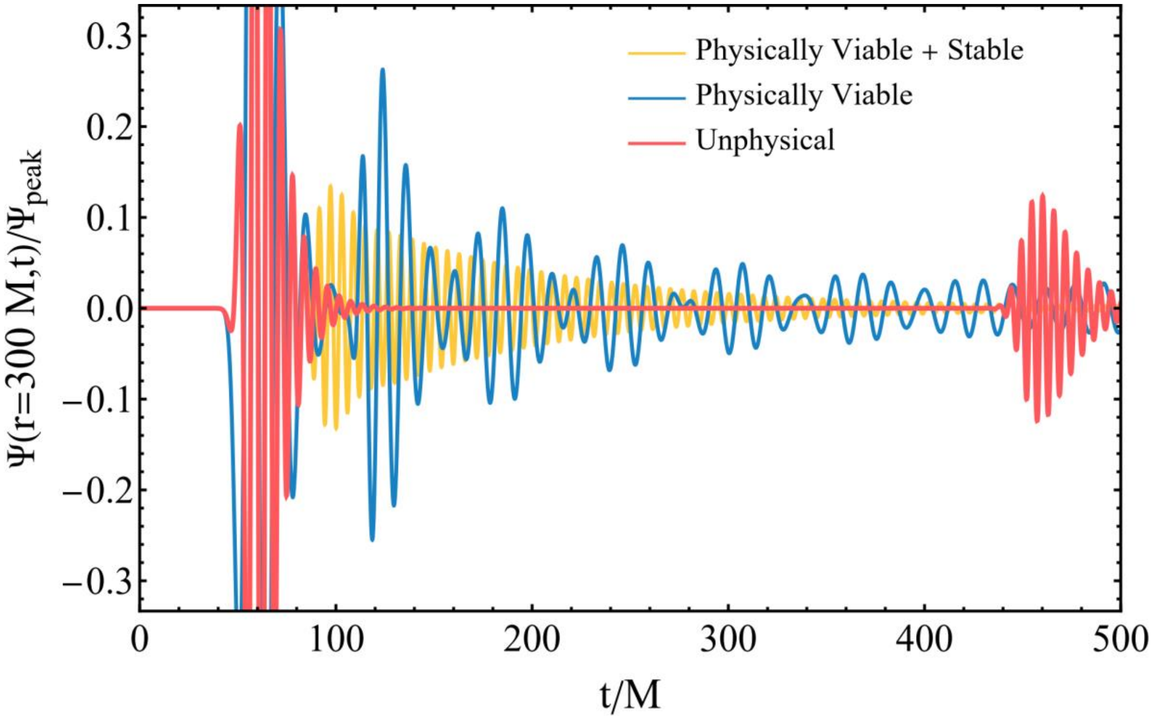
The potential barrier is not at surface but within the compact object (centrifugal barrier);

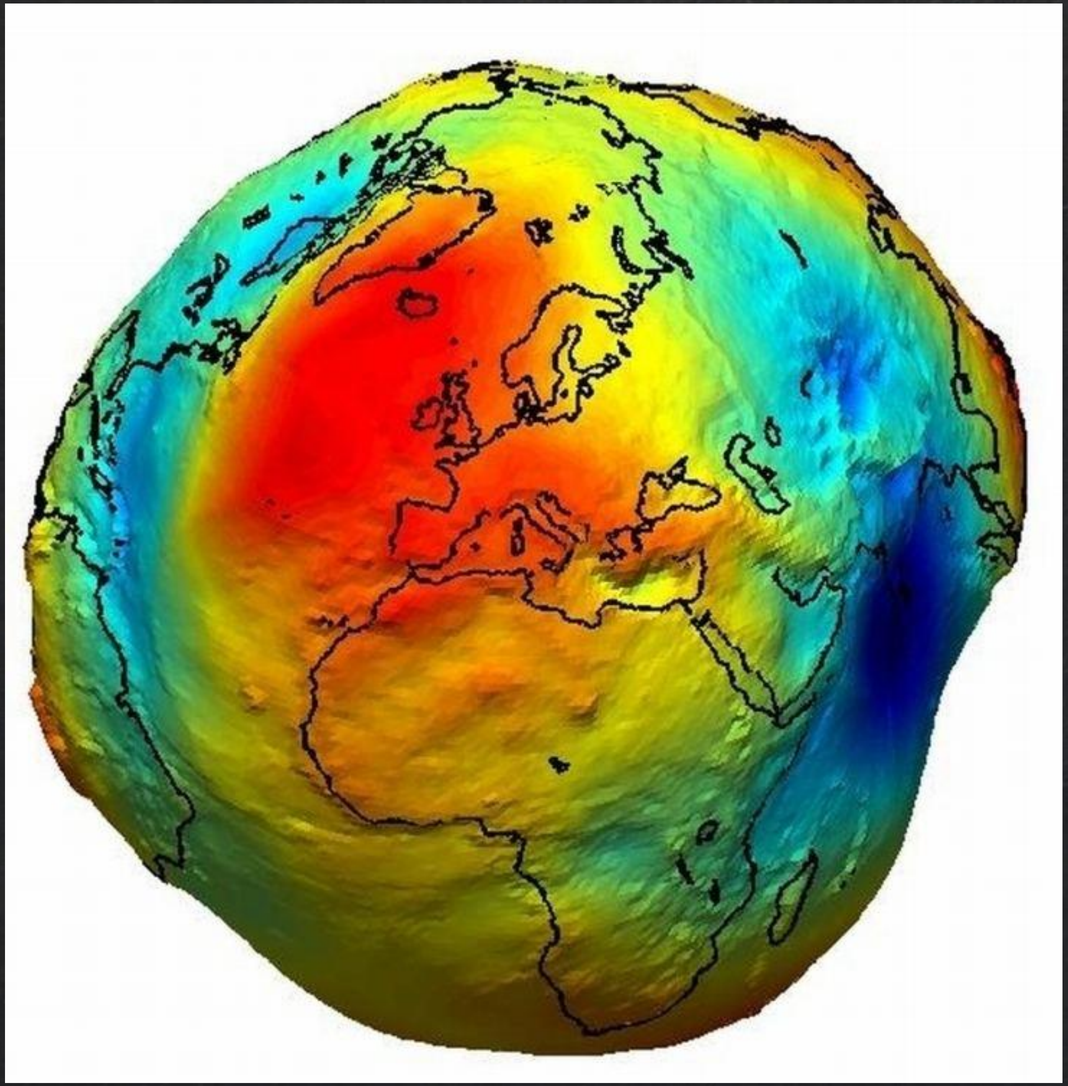
Perturbation takes more time traveling within the star than outside. Much longer time between echoes.



Echoes Ultracompact Stars







Multipole Moments
And
Tidal Effects

Multipole Moments

In Newtonian Gravity:

$$\Phi(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{M_{\ell m}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \varphi) \,,$$

In GR:

More complex definition, but similar idea (in ACMC coordinates)

Multipole Moments

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Black Holes have no hair…

$$M_\ell^{\rm BH} + i S_\ell^{\rm BH} = M^{\ell+1} (i\chi)^\ell$$

Axisymmetric & Equatorially Symmetric

In GR:

More complex definition, but similar idea (in ACMC coordinates)

… but ECOs can

$$M_{\ell m}^{\rm ECO} = M_\ell^{\rm BH} + \delta M_{\ell m}$$

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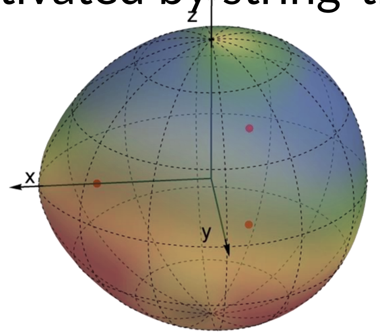
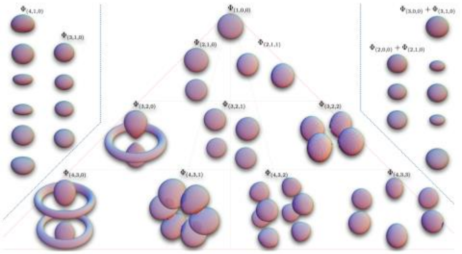
Nonspherical ECOs?

Microstate geometries

Multi-center solutions motivated by string-theory

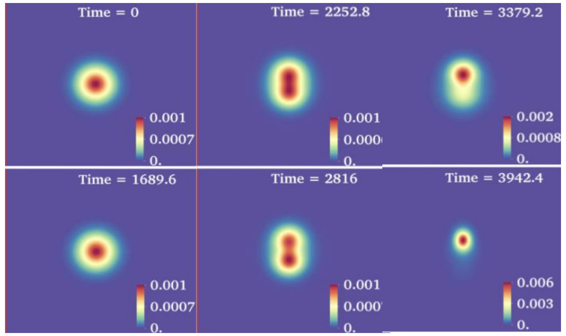


[Raposo+, 2007.01743; Bena, Mayerson, 2006.10750]



Multipolar boson stars

[Herdeiro+,2008.10608]



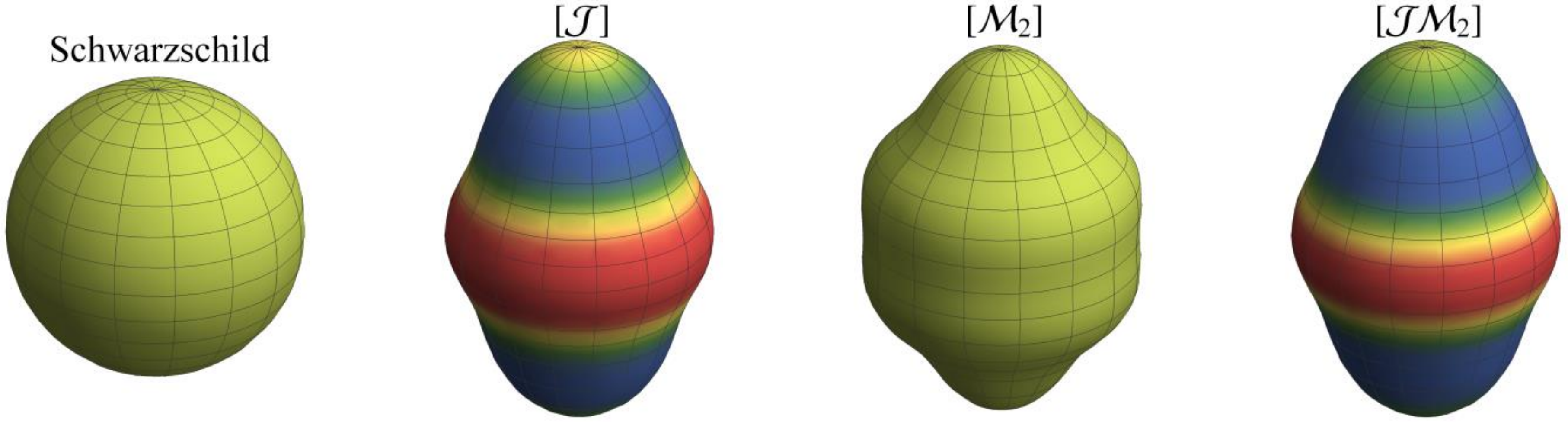
Prolate Proca stars

[Herdeiro+,2311.14800]

[Etevaldo's talk]

Fundamental state of Proca star is Prolate!

Soft ECOs



$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \sum_{n=1}^{\infty} \epsilon^n h^{(n)}_{\mu\nu}$$

$$h^{(n)}_{\mu\nu} = \begin{pmatrix} -f H_0^{n\ell} P_\ell & 0 & 0 & h_0^{n\ell} P'_\ell \\ 0 & f^{-1} H_2^{n\ell} P_\ell & 0 & 0 \\ 0 & 0 & r^2 K^{n\ell} P_\ell & 0 \\ h_0^{n\ell} P'_\ell & 0 & 0 & r^2 \sin^2\theta K^{n\ell} P_\ell \end{pmatrix} .$$

Soft ECO condition:
Curvature at surface like that of horizon.

$$\mathcal{K}^{1/2} \sim 1/\mathcal{M}^2$$

The multipolar deviations vanish logarithmically (or faster)

$$M_\ell \sim a/\log \Delta$$

Fuzzballs

Microstate geometries

Multi-center solutions motivated by string-theory

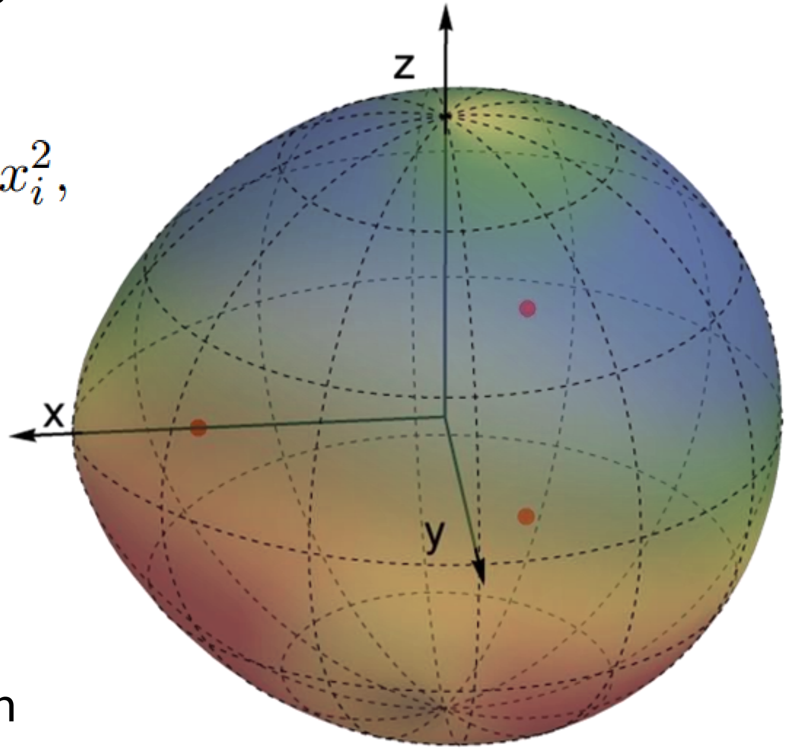
$$ds^2 = -e^{2U}(dt+\omega)^2 + e^{-2U}\sum_{i=1}^{3} dx_i^2,$$

Where U is a combination of:

$$V = v_0 + \sum_{a=1}^{N} \frac{v_i}{r_a}, \quad L_I = \ell_{0I} + \sum_{a=1}^{N} \frac{\ell_{I,a}}{r_a},$$

$$K^I = k_0^I + \sum_{a=1}^{N} \frac{k_a^I}{r_a}, \quad M = m_0 + \sum_{a=1}^{N} \frac{m_a}{r_a}.$$

Since this are harmonic functions, metric is in ACMC form:



Multipolar Structure of Fuzzball

$$\delta M_{\ell m} \sim a L^n$$

[Raposo+, 2007.01743; Bena, Mayerson, 2006.10750]

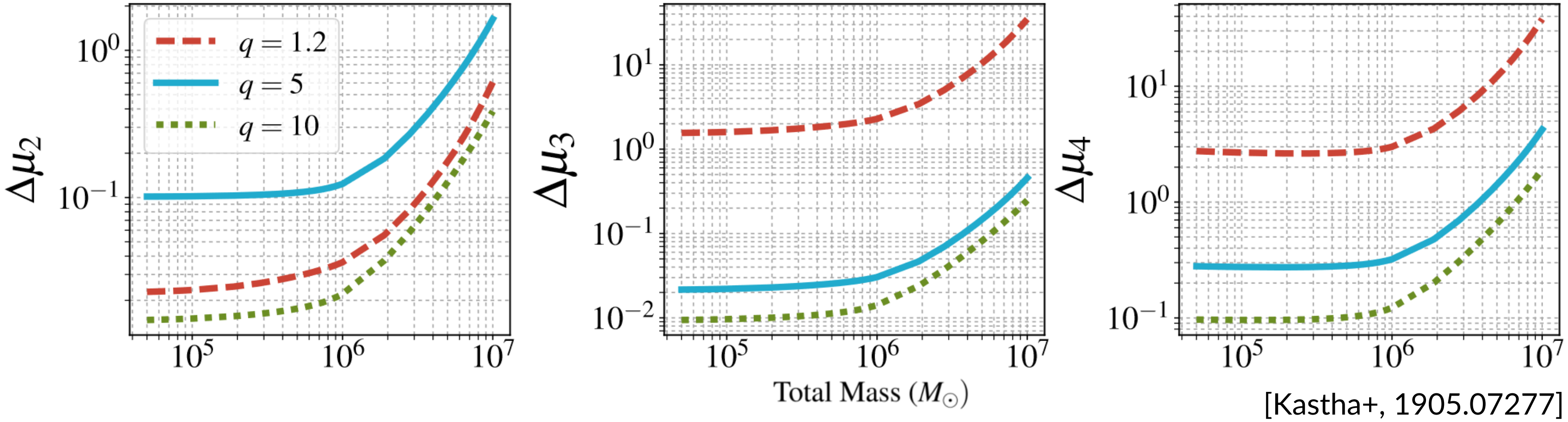
Multipole Moments

The multipole moments affect the phase of the gravitational wave (inspiral).

The dominant term appears at 2PN order. $$\psi_{\ell=2} = \frac{75}{64}\frac{\left(m_2 M_2^{(1)} + m_1 M_2^{(2)}\right)}{(m_1 m_2)^2}\frac{1}{v}$$

However: Correlated with the spins (not measured accurately so far).

For LISA: EMRI are a gold signal for multipolar tests. Can constrain a large set of multipoles!



[Kastha+, 1905.07277]

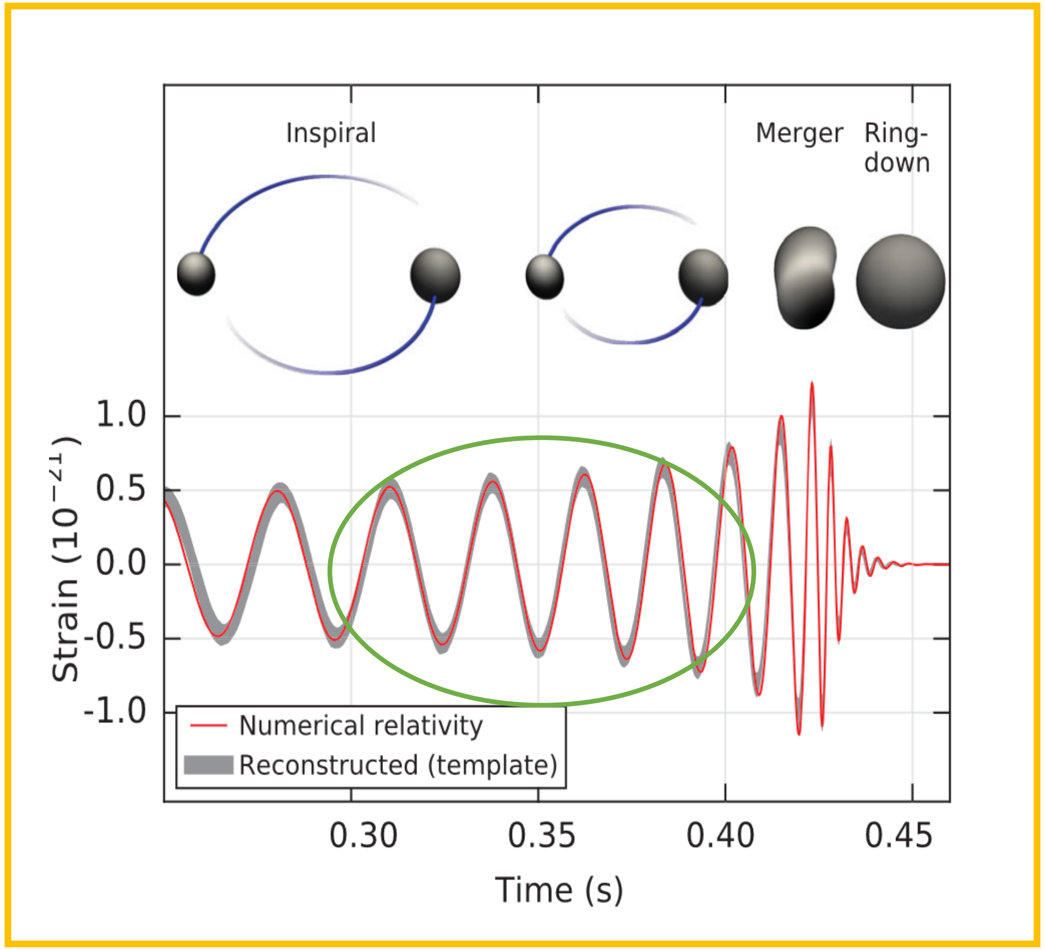
A detection of EMRI can potentially allow to constrain M_2 up to one part in 10^4

Tidal Love numbers

For BHs:

$$M/R = 0.5$$

$$k_2 = 0$$



Tidal Love numbers

For BHs:

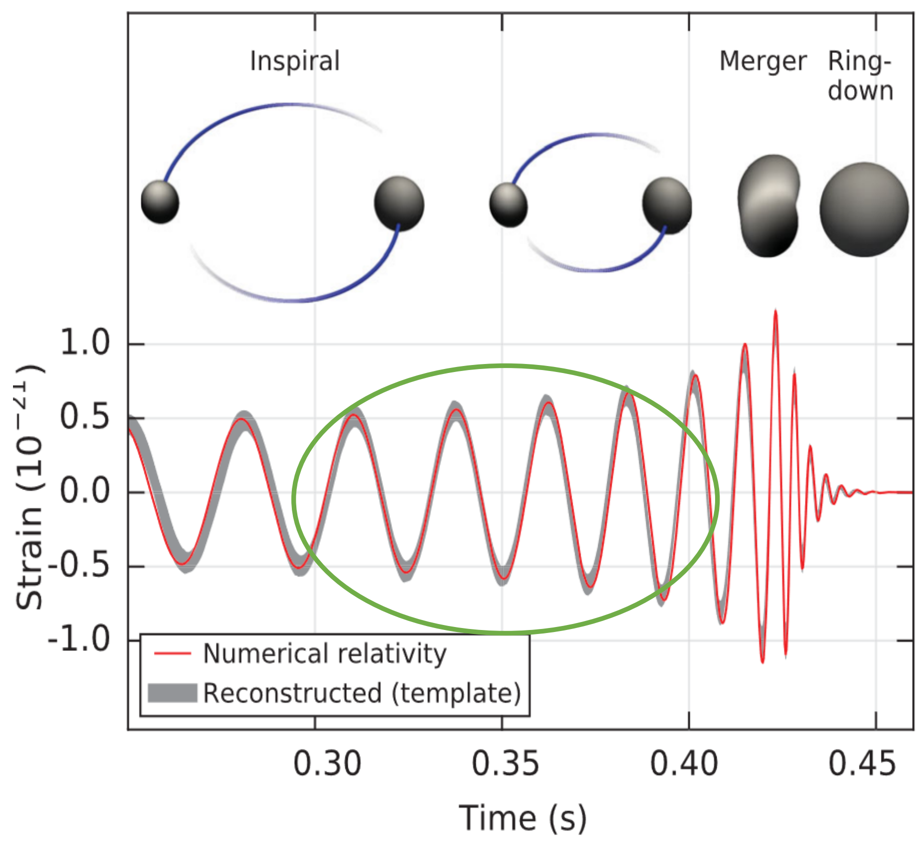
$$M/R = 0.5$$

$$k_2 = 0$$

For NSs:

$$M/R \sim [0.1, 0.2]$$

$$k_2 \neq 0 \sim O(100)$$



Tidal Love numbers

For BHs:

$$M/R = 0.5$$

$$k_2 = 0$$

For NSs:

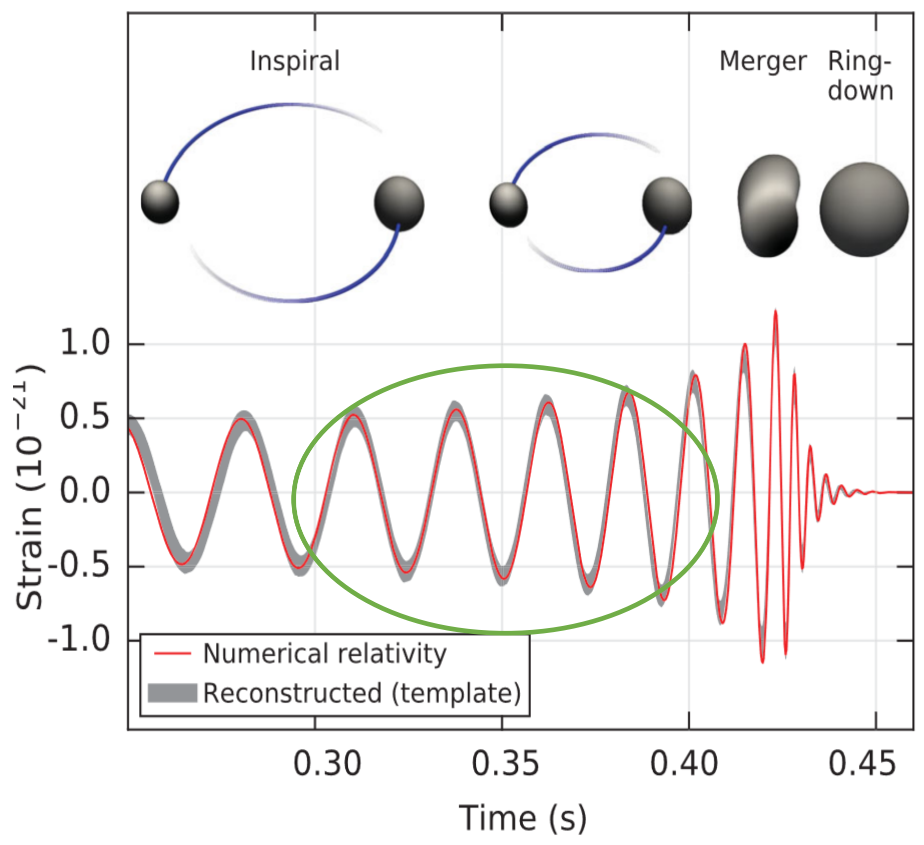
$$M/R \sim [0.1, 0.2]$$

$$k_2 \neq 0 \sim O(100)$$

For ECOs:

$$M/R < 0.5 \qquad k_2 \neq 0$$

$$M/R \to 0.5 \qquad k_2 \to 0$$



For Hard ECOs

Tidal Love numbers: ECOs

- Tidal Love number vanishes logarithmically in the BH limit.

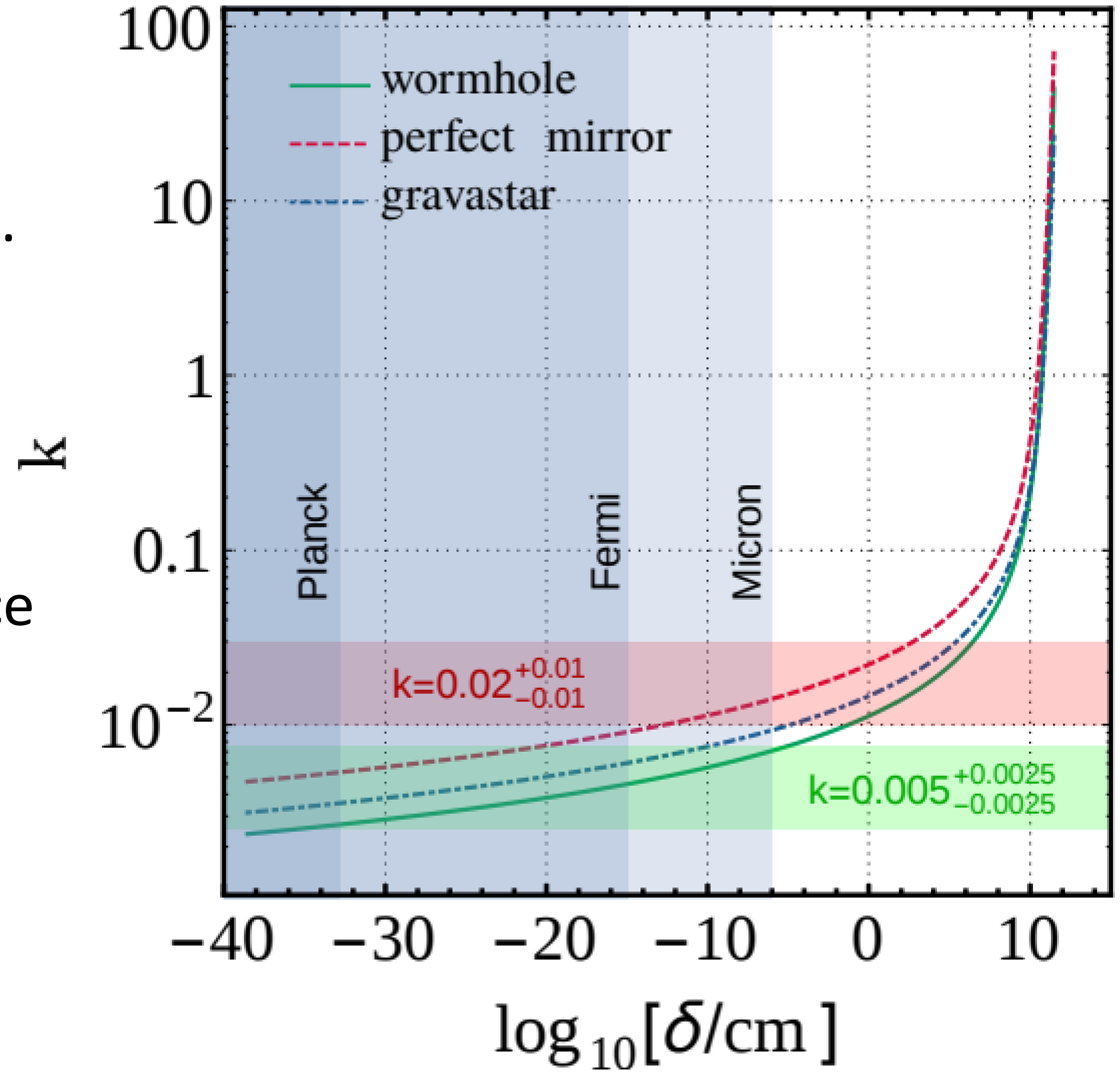
$$k_2 \sim \log^{-1} \delta$$

- The Love number can be converted into a distance of ECO surface from horizon!

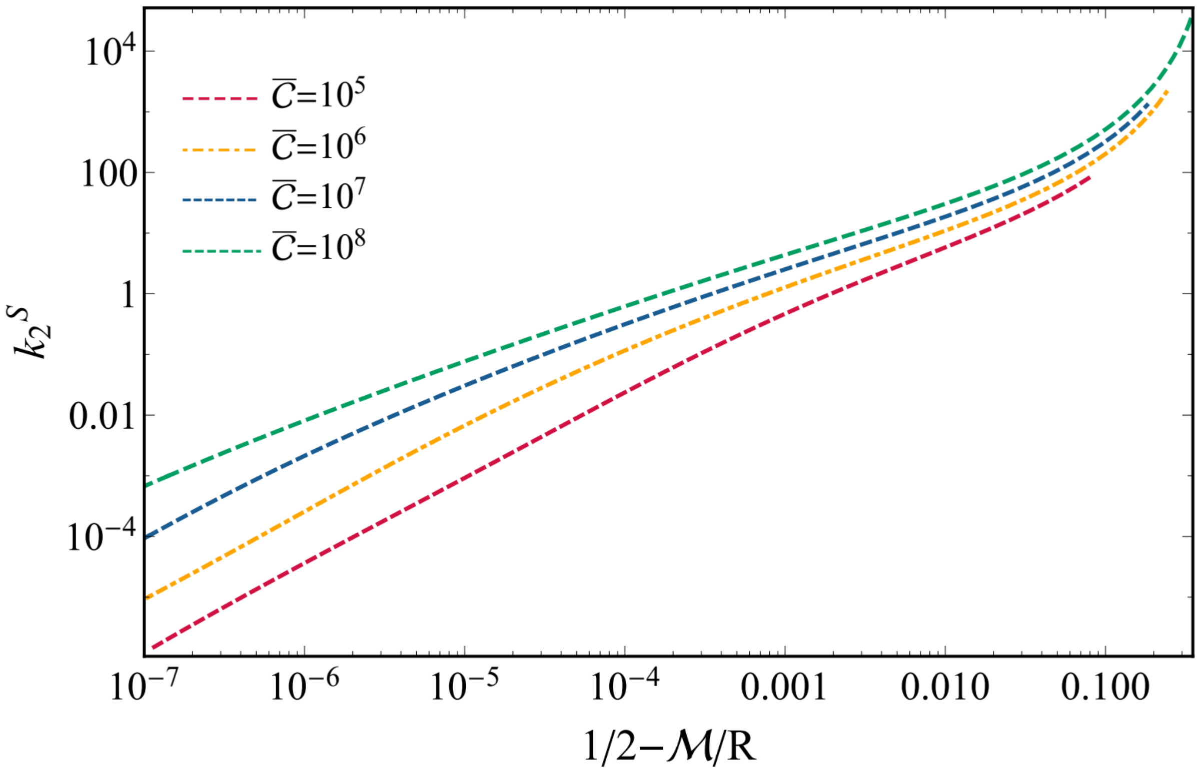
$$\delta \sim 2Me^{-1/k_2}$$

- Possible to probe Planckian corrections to the horizon!

$$k_2 \sim 10^{-2} \rightarrow \delta \sim 10^{-33}\text{cm}$$



Tidal Love numbers: ECOs



For Ultracompact Stars:

- Tidal Love number vanishes polynomially in the BH limit.

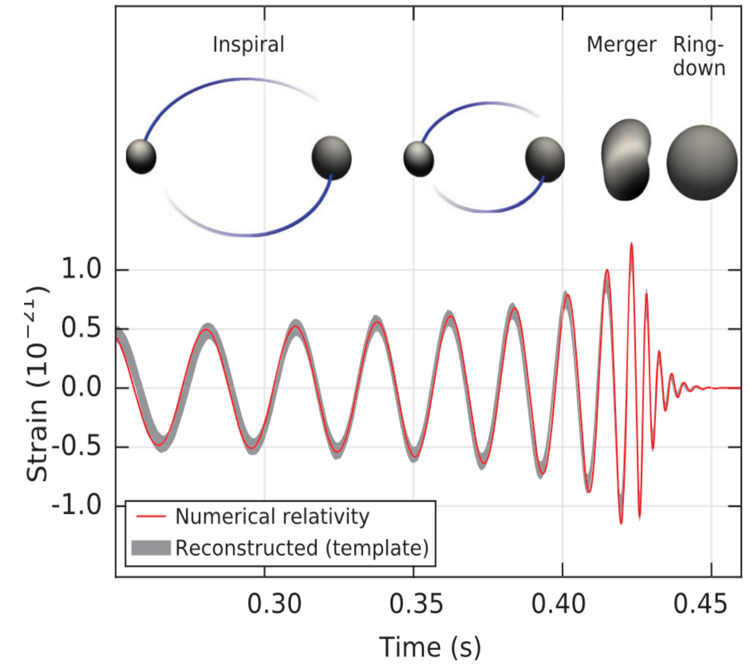
$$k_2 \sim \delta^{\alpha}$$

- For star-like ECOs it may be challenging to measure Planckian corrections to the horizon structure.

Love numbers in the waveform

Tidal effects add linearly to the waveform phase.

$$h \sim \mathcal{A} e^{i(\psi_{\rm PP} + \psi_{\rm TD})}$$



$$\psi_{\rm PP} \propto \left[1 - \left(\frac{743}{336} + \frac{11}{4}\nu\right) \frac{(m\pi f)^{2/3}}{c^2} + \ldots\right]$$

Newtonian

1PN

etc

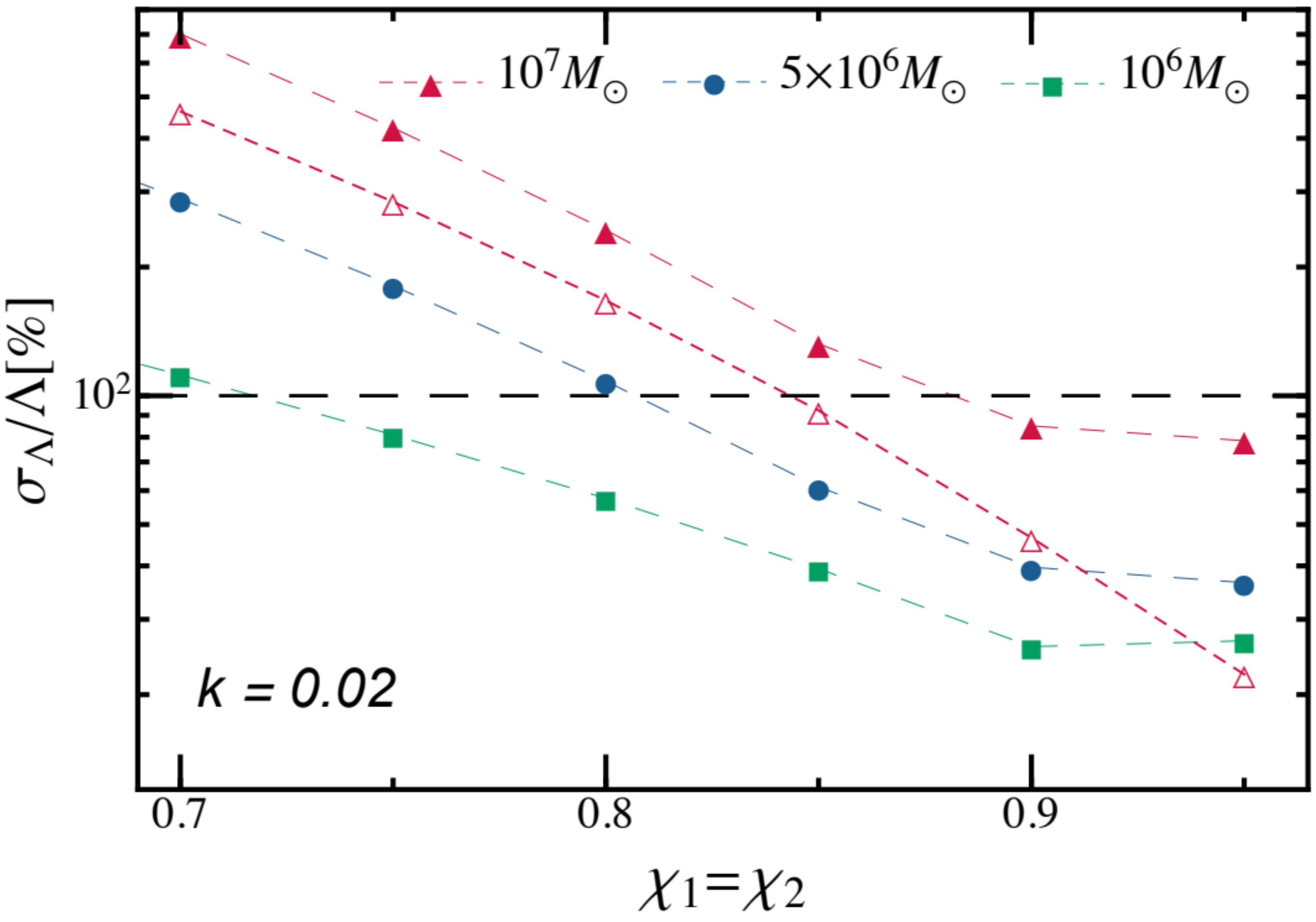
$$\psi_{\rm TD} = -\psi_N \frac{625}{m^5} \Lambda \left(\frac{v}{c}\right)^{10}$$

5PN contribution!

Average tidal deformability

Detectability

[Maselli+, 2018]



Acknowledgments

Thank you all for attending!

Questions?

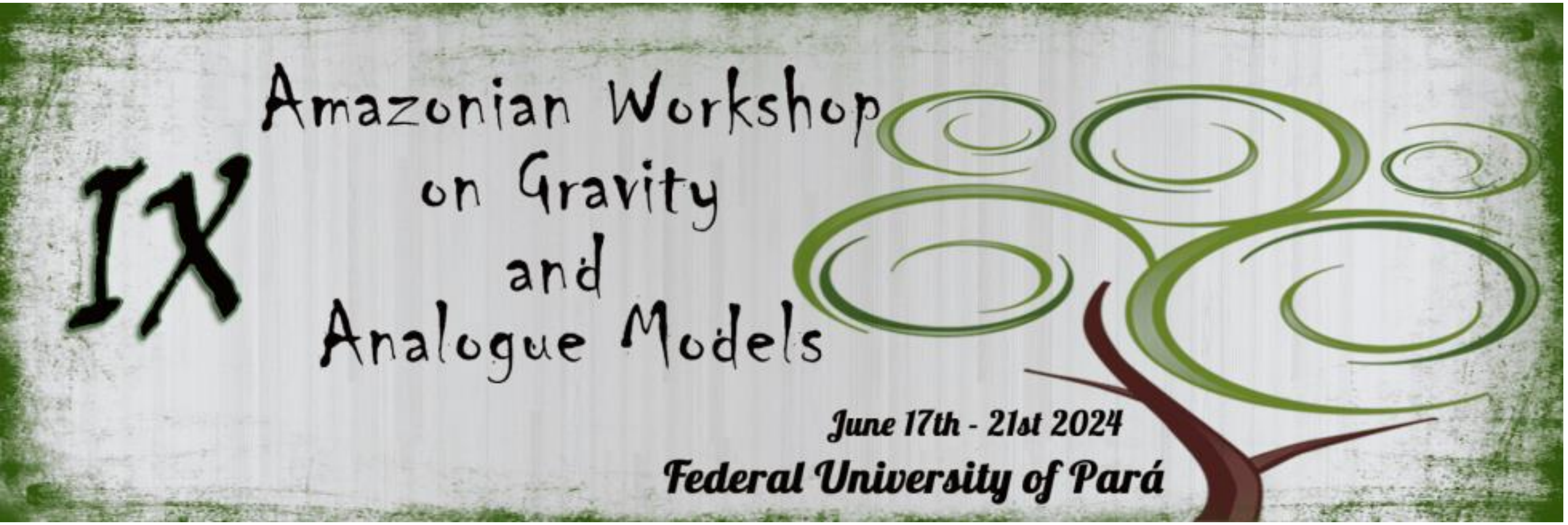
I want to thank the support of:

CIDMA

fct Fundação para a Ciência e a Tecnologia

MARIE CURIE ACTIONS

Compact Objects and How to Model Them
Part III



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